2004

7 (c) Given that $x = \dfrac{e^{2y}-1}{e^{2y}+1}$,

(i) show that $e^{2y} = \dfrac{1+x}{1-x}$

multiply both Sides by LCM.

$$x(e^{2y}+1) = e^{2y}-1$$

expand

$$xe^{2y} + x = e^{2y} - 1$$

Bring "e^{2y}" terms to LHS

$$xe^{2y} - e^{2y-1} = -x - 1$$

factorise (HCF)

$$e^{2y}(x-1) = -x-1$$

Divide by (x-1)

$$e^{2y} = \frac{-x-1}{x-1}$$

Change all signs in fraction

$$e^{2y} = \frac{1+x}{1-x}$$

QED

Implicit differentiation

need to differentiate both sides

(ii) show that $\dfrac{dy}{dx}$ can be expressed in the form $\dfrac{p}{1-x^p}$, $p, q \in \mathbf{N}$.

$$e^{2y} = \frac{1+x}{1-x}$$

LHS use implicit differentiation

$$e^{ax} \rightarrow ae^{ax}$$

$$e^{2y} \rightarrow 2e^{2y}\frac{dy}{dx}$$

this is added on because of y term

RHS use Quotient rule

$$\frac{u}{v} \rightarrow \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

careful with signs!

$$\frac{1+x}{1-x} \rightarrow \frac{(1-x)(1) - (1+x)(-1)}{(1-x)^2} = \frac{1 - \cancel{x} + 1 + \cancel{x}}{(1-x)^2} = \frac{2}{(1-x)^2}$$

Derivative

LHS = RHS

$$\cancel{2}e^{2y}\frac{dy}{dx} = \frac{\cancel{2}\,1}{(1-x)^2}$$

* But $e^{2y} = \dfrac{1+x}{1-x}$

$$\Rightarrow {}^{*}\left(\frac{1+x}{1-x}\right)\frac{dy}{dx} = \frac{1}{(1-x)^2}$$

multiply by (1 - x)

divide by (x + 1)

(1 - x)'s cancel

$$\frac{dy}{dx} = \left(\frac{\cancel{1-x}}{1+x}\right)\left(\frac{1}{(1-x)^{\cancel{2}}}\right)$$

$$= {}^{*}\frac{1}{(1+x)(1-x)} = \frac{1}{1-x^2}$$

* Difference of 2 Squares