Question 9 **(50 marks)**

The *atmospheric pressure* is the pressure exerted by the air in the earth's atmosphere. It can be measured in kilopascals (kPa). The average atmospheric pressure varies with altitude: the higher up you go, the lower the pressure is.

Some students are investigating this variation in pressure, using some data that they found on the internet. They have information about the average pressure at various altitudes.

Six of the entries in the data set are as shown in the table below:

altitude (km)	0	1	2	3	4	5
pressure (kPa)	101·3	89·9	79·5	70·1	61·6	54·0

By looking at the pattern, the students are trying to find a suitable model to match the data.

(a) Hannah suggests that this is approximately a geometric sequence. She says she can match the data fairly well by taking the first term as 101·3 and the common ratio as 0·883.

(i) Complete the table below to show the values given by Hannah's model, correct to one decimal place.

altitude (km)	0	1	2	3	4	5
pressure (kPa)	101·3	89·4	79·0	69·7	61·6	54·4
		–0·6%	–0·6%	–0·6%	0%	+0·7%

(ii) By considering the percentage errors in the above values, insert an appropriate number to complete the statement below.

"Hannah's model is accurate to within __1__ %."

(b) Thomas suggests modelling the data with the following exponential function:

$$p = 101{\cdot}3 \times e^{-0{\cdot}1244h}$$

where p is the pressure in kilopascals, and h is the altitude in kilometres.

(i) Taking any **one** value other than 0 for the altitude, verify that the pressure given by Thomas's model and the pressure given by Hannah's model differ by less than 0·01 kPa.

Altitude 1: $101{\cdot}3e^{-0{\cdot}1244} - 101{\cdot}3(0{\cdot}883) \approx 0{\cdot}0027 < 0{\cdot}01$

Altitude 2: $101{\cdot}3e^{-0{\cdot}1244\times 2} - 101{\cdot}3(0{\cdot}883)^2 \approx 0{\cdot}0048 < 0{\cdot}01$

Altitude 3: $101{\cdot}3e^{-0{\cdot}1244\times 3} - 101{\cdot}3(0{\cdot}883)^3 \approx 0{\cdot}0063 < 0{\cdot}01$

Altitude 4: $101{\cdot}3e^{-0{\cdot}1244\times 4} - 101{\cdot}3(0{\cdot}883)^4 \approx 0{\cdot}0074 < 0{\cdot}01$

Altitude 5: $101{\cdot}3e^{-0{\cdot}1244\times 5} - 101{\cdot}3(0{\cdot}883)^5 \approx 0{\cdot}0081 < 0{\cdot}01$

(ii) Explain how Thomas might have arrived at the value of the constant 0·1244 in his model.

He might have assumed it was of the form $101{\cdot}3e^{-kt}$ and then used one of the observations to find k.

He might have put various values of t into $p(t) = 101{\cdot}3e^{-kt}$, and found an average of the resulting values of k.

He might have got the natural log of the ratio of consecutive terms.

He might have plotted the log of the pressure against the altitude and used the slope of the best-fit line to find k.

(c) Hannah's model is *discrete*, while Thomas's is *continuous*.

(i) Explain what this means.

Hannah's model gives values for the pressure at separate (whole number) values for the altitude.

Thomas's model gives a value for the pressure at any real value of the altitude, whether it's a whole number or not.

(ii) State one advantage of a continuous model over a discrete one.

You are not restricted to the specific discrete values of the independent variable; you can also work with values between any two given values – any value you like.

(d) Use Thomas's model to estimate the atmospheric pressure at the altitude of the top of Mount Everest: 8848 metres.

$$p(8{\cdot}848) = 101{\cdot}3e^{-0{\cdot}1244 \times 8{\cdot}848} \approx 33{\cdot}7 \text{ kPa}.$$

(e) Using Thomas's model, find an estimate for the altitude at which the atmospheric pressure is half of its value at sea level (altitude 0 km).

$$p(h) = \tfrac{1}{2}\,p(0)$$
$$101{\cdot}3e^{-0{\cdot}1244h} = \tfrac{1}{2}(101.3)$$
$$e^{0{\cdot}1244h} = 2$$
$$0{\cdot}1244h = \ln 2$$
$$h \approx 5{\cdot}57 \text{ km}$$

(f) People sometimes experience a sensation in their ears when the pressure changes. This can happen when travelling in a fast lift in a tall building. Experiments indicate that many people feel such a sensation if the pressure changes rapidly by 1 kilopascal or more. Suppose that such a person steps into a lift that is close to sea level. Taking a suitable approximation for the distance between two floors, estimate the number of floors that the person would need to travel in order to feel this sensation.

$$p(h) = 100{\cdot}3$$
$$101{\cdot}3e^{-0{\cdot}1244h} = 100{\cdot}3$$
$$e^{-0{\cdot}1244h} = \frac{100{\cdot}3}{101{\cdot}3}$$
$$-0{\cdot}1244h = \ln\left(\frac{100{\cdot}3}{101{\cdot}3}\right)$$
$$h = \frac{1}{0{\cdot}1244}\left(\ln 101{\cdot}3 - \ln 100{\cdot}3\right)$$
$$h \approx 0{\cdot}0797 \text{ km} \approx 80 \text{ m}$$

Allowing 3 m per floor, it's about 27 floors.