Q1	Model Solution – 25 Marks	Marking Notes
(a)	$2\left(x^2 - \frac{7}{2}x - 5\right)$ $= 2\left(\left(x - \frac{7}{4}\right)^2 - \frac{129}{16}\right)$ $= 2\left(\left(x - \frac{7}{4}\right)^2\right) - \frac{129}{8}$	**Scale 5D (0, 2, 3, 4, 5)** *Low Partial Credit:* • $a = 2$ identified explicitly or as factor *Mid partial Credit:* • Completed square *High partial Credit:* • h or k identified from work
(b)	$\left(\frac{7}{4}, \quad \frac{-129}{8}\right)$	**Scale 10B (0, 4, 10)** *Partial Credit:* • One relevant co-ordinate identified

(c) (i)	$f(x)$ has min point as $a > 0$ y co-ordinate of min < 0 ⇒graph must cut x-axis twice hence two real roots. **or** $b^2 - 4ac = 49 + 80 > 0$ Therefore real roots	**Scale 5B (0, 3, 5)** *Partial Credit:* • Mention of $a > 0$ • $b^2 - 4ac$ • Identifies location of one or two roots, e.g. between 4 and 5.
c (ii)	$2x^2 - 7x - 10 = 0$ $2\left(\left(x - \frac{7}{4}\right)^2\right) - \frac{129}{8} = 0$ $\left(x - \frac{7}{4}\right)^2 = \frac{129}{16}$ $x - \frac{7}{4} = \pm\frac{\sqrt{129}}{4}$ $x = \frac{7}{4} \pm \sqrt{\frac{129}{16}}$ **OR** $2x^2 - 7x - 10 = 0$ $x = \frac{7 \pm \sqrt{49 + 80}}{4}$ $= \frac{7 \pm \sqrt{129}}{4}$ $x = \frac{7}{4} \pm \sqrt{\frac{129}{16}}$	**Scale 5C (0, 3, 4, 5)** *Low Partial Credit:* • Formula with some substitution • Equation rewritten with some transpose *High Partial Credit:* • $x - \frac{7}{4} = \pm\frac{\sqrt{129}}{4}$ or equivalent

2 (2015)

Question 2 **(25 marks)**

$f(x)=x^3-3x^2-9x+11$

$f(1)=1^3-3(1)^2-9+11=0$

$\Rightarrow x=1$ is a solution.

$(x-1)$ is a factor

$$
\begin{array}{r|l}
 & x^2-2x \quad -11 \\
\hline
x-1 & x^3-3x^2-9x+11 \\
 & x^3-x^2 \\
 & \quad -2x^2-9x+11 \\
 & \quad -2x^2+2x \\
 & \qquad\quad -11x+11 \\
 & \qquad\quad -11x+11
\end{array}
$$

or

$(x-1)(x^2+Ax-11)=x^3-3x^2-9x+11$

$\Rightarrow x^3+Ax^2-x-x^2-Ax+1=x^3-3x^2-9x+11$

$\Rightarrow A-1=-3$

$\Rightarrow A=-2$

or

	x^2	$-2x$	-11
x	x^3	$-2x^2$	$-11x$
-1	$-x^2$	$2x$	11

Hence, other factor is $x^2-2x-11$

$$x=\frac{2\pm\sqrt{(-2)^2-4(1)(-11)}}{2(1)}=\frac{2\pm\sqrt{48}}{2}=\frac{2\pm4\sqrt{3}}{2}=1\pm2\sqrt{3}$$

Solutions: $\{1,\ 1+2\sqrt{3}\ ,\ 1-\ 2\sqrt{3}\ \}$

3 (2012)

(a) The complex numbers z_1, z_2 and z_3 are such that $\frac{2}{z_1} = \frac{1}{z_2} + \frac{1}{z_3}$, $z_2 = 2 + 3i$ and $z_3 = 3 - 2i$, where $i^2 = -1$. Write z_1 in the form $a + bi$, where $a, b \in \mathbb{Z}$.

$$\frac{2}{z_1} = \frac{1}{z_2} + \frac{1}{z_3} = \frac{1}{2+3i} + \frac{1}{3-2i}$$

$$= \frac{3-2i+2+3i}{(2+3i)(3-2i)} = \frac{5+i}{12+5i}$$

$$\Rightarrow \frac{z_1}{2} = \frac{12+5i}{5+i}$$

$$= \frac{12+5i}{5+i} \times \frac{5-i}{5-i}$$

$$= \frac{65+13i}{26}$$

$$\Rightarrow z_1 = 5+i$$

or

$$\frac{1}{2+3i} = \frac{1}{2+3i} \cdot \frac{2-3i}{2-3i} = \frac{2-3i}{4+9} = \frac{2-3i}{13}$$

$$\frac{1}{3-2i} = \frac{1}{3-2i} \cdot \frac{3+2i}{3+2i} = \frac{3+2i}{4+9} = \frac{3+2i}{13}$$

$$\frac{1}{2+3i} + \frac{1}{3-2i} = \frac{2-3i}{13} + \frac{3+2i}{13} = \frac{5-i}{13}$$

$$\frac{2}{z_1} = \frac{5-i}{13}$$

Let $z_1 = a + bi$

$$\frac{2}{a+bi} = \frac{5-i}{13}$$

$$26 = (5-i)(a+bi)$$

$$26 + (0)i = 5a + 5bi - ai + b$$

$$26 + (0)i = (5a+b) + (-a+5b)i$$

$$\Rightarrow 5a+b=26 \ \ldots\text{(i) and } \ -a+5b=0 \ \ldots\text{(ii)}$$

$$\begin{aligned} &\text{(i):} & 5a+b&=26 \\ &\text{(ii):} & -5a+25b&=0 \\ \hline & & 26b&=26 \\ & & b&=1 \end{aligned}$$

From (ii): $5b=a$

$$\Rightarrow a=5$$

$$z_1=5+i$$

5 (2014)

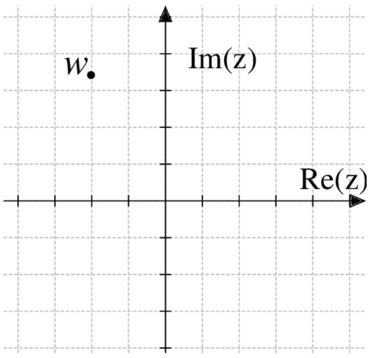
(a) $w=-1+\sqrt{3}i$ is a complex number, where $i^2=-1$.

(i) Write w in polar form.

We have $|w|=\sqrt{(-1)^2+\sqrt{3}^2}=\sqrt{4}=2$. Also, if $\arg(w)=\theta$, then $\tan(\theta)=\frac{\sqrt{3}}{-1}=-\sqrt{3}$ and θ lies in the second quadrant (from the diagram). Therefore $\theta=\tan^{-1}(-\sqrt{3})=\frac{2\pi}{3}$radians. So

$$w=2\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$$

MODEL ANSWER BY thematthstutor.ie ONLINE SUPPORT SYSTEM FOR PROJECT MATHS



(ii) Use De Moivre's Theorem to solve the equation $z^2=-1+\sqrt{3}i$. Give your answer(s) in rectangular form.

Suppose that the polar form of z is given by $z = r(\cos\theta + i\sin\theta)$. Then

$$[r(\cos\theta + i\sin\theta)]^2 = 2(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3})$$

By De Moivre's Theorem this is equivalent to

$$r^2(\cos(2\theta) + i\sin(2\theta)) = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right).$$

Therefore $r^2 = 2$ and $2\theta = \frac{2\pi}{3} + 2n\pi$, $n \in \mathbb{Z}$. So $r = \sqrt{2}$ and $\theta = \frac{\pi}{3} + n\pi$. We get two distinct solutions (corresponding to $n = 0$ and $n = 1$).

$$z_1 = \sqrt{2}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = \sqrt{2}\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = \frac{1}{\sqrt{2}} + i\sqrt{\frac{3}{2}}$$

and

$$z_2 = \sqrt{2}\left(\cos(\frac{\pi}{3} + \pi) + i\sin(\frac{\pi}{3} + \pi)\right) = \sqrt{2}\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = -\frac{1}{\sqrt{2}} - i\sqrt{\frac{3}{2}}$$

MODEL ANSWER BY thematshtutor.ie
ONLINE SUPPORT SYSTEM FOR PROJECT MATHS

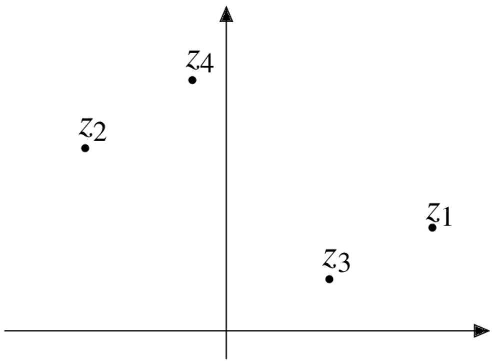
(b) Four complex numbers z_1, z_2, z_3 and z_4 are shown on the Argand diagram. They satisfy the following conditions:

$$\begin{aligned} z_2 &= iz_1 \\ z_3 &= kz_1, \text{ where } k \in \mathbb{R} \\ z_4 &= z_2 + z_3 \end{aligned}$$

The same scale is used on both axes.

(i) Identify which number is which by labelling the points on the diagram.

(ii) Write down the approximate value of k.



Answer: $\frac{1}{2}$

Explanation: Multiplication by i rotates a complex number by 90° anticlockwise about the origin - so z_2 is obtained by rotating z_1 through 90° about the origin.

Since $z_3 = kz_1$, we must have 0, z_1 and z_3 being collinear.

Since $z_4 = z_2 + z_3$, we must have 0, z_2, z_4 and z_3 forming a parallelogram.

MODEL ANSWER BY thematshtutor.ie
ONLINE SUPPORT SYSTEM FOR PROJECT MATHS

$$w^4 = \frac{81}{16}\left(\cos\left(\frac{4\pi}{9} + 2n\pi\right) + i\sin\left(\frac{4\pi}{9} + 2n\pi\right)\right)$$

$$w = \frac{3}{2}\left(\cos\left(\frac{\pi}{9} + \frac{n\pi}{2}\right) + i\sin\left(\frac{\pi}{9} + \frac{n\pi}{2}\right)\right), \quad n = 0, 1, 2, 3.$$

$$w = \frac{3}{2}\left(\cos\left(\frac{\pi}{9}\right) + i\sin\left(\frac{\pi}{9}\right)\right), \quad \frac{3}{2}\left(\cos\left(\frac{11\pi}{18}\right) + i\sin\left(\frac{11\pi}{18}\right)\right),$$

$$\frac{3}{2}\left(\cos\left(\frac{10\pi}{9}\right) + i\sin\left(\frac{10\pi}{9}\right)\right), \quad \frac{3}{2}\left(\cos\left(\frac{29\pi}{18}\right) + i\sin\left(\frac{29\pi}{18}\right)\right)$$