6th year honours maths
Test on Algebra Chapter 2

Show that the roots of the equation $x^2 - (a + d)x + (ad - b^2) = 0$ are real.

use discriminant

$\Delta = b^2 - 4ac$

$\Delta \geq 0$ when Roots are Real

must recognise that

$a^2 - 2ad + d^2 = (a-d)^2$

$$\Delta = (-(a+d))^2 - 4(1)(ad - b^2)$$
$$= a^2 + 2ad + d^2 - 4ad + 4b^2$$
$$= a^2 - 2ad + d^2 + 4b^2$$
$$= (a-d)^2 + 4b^2 \geq 0$$

Since the square of a number ≥ 0

$\Rightarrow$ Roots are real

A section of the graph of a polynomial

$$f(x) = ax^3 + bx^2 + cx + d$$

is drawn in this diagram.

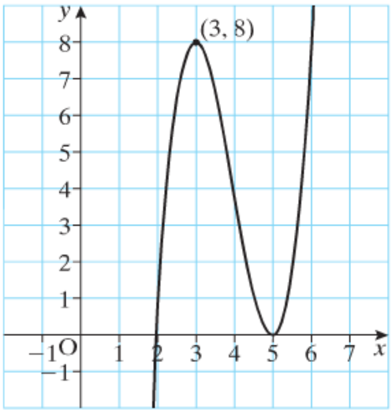
(i) Find the roots of this polynomial.

(ii) Write an expression for $f(x)$ in terms of the factors of this polynomial.

(iii) Find the values of a, b, c and d.

(iv) Find an expression for the reflected image of this curve in the x-axis.

(v) Find an expression for the reflected image of this curve in the y-axis.



Roots = x-intercepts

(i) $x = 2, 5, 5$

n-multiplier

(ii) $f(x) = n(x-2)(x-5)(x-5)$

$f(3) = 8$

$f(3) = n(3-2)(3-5)(3-5) = 8$

$\Rightarrow n(1)(-2)(-2) = 8$

$4n = 8$

$n = 2$

$\Rightarrow f(x) = 2(x-2)(x-5)(x-5)$

expand

(iii) $f(x) = 2(x-2)(x-5)(x-5)$

$= (2x-4)(x^2 - 10x + 25)$

$= 2x^3 - 20x^2 + 50x - 4x^2 + 40x - 1$

$= 2x^3 - 24x^2 + 90x - 100$

$f(x) = ax^3 + bx^2 + cx + d$

$\Rightarrow a = 2 \quad b = -24 \quad c = 90 \quad d = -100$

Reflected image in x-axis is -f(x)

(iv) $-f(x) = -2x^3 + 24x^2 - 90x + 100$

Reflected image in y-axis = $g(x)$

(v) Roots would be: $-2, -5, -5$

factors $(x+2)(x+5)(x+5)$

shape would be negative but size same $\Rightarrow$ multiplier $= -2$

$g(x) = -2(x+2)(x+5)(x+5)$

project maths paper 2012

A cubic function f is defined for $x \in \mathbb{R}$ as

$$f : x \mapsto x^3 + (1 - k^2)x + k, \quad \text{where } k \text{ is a constant.}$$

Factor THEOREM

$f(-k) = 0$ if $-k$ is Root

(a) Show that $-k$ is a root of f.

Is $(-k)^3 + (1-k^2)(-k) + k = 0$

$k^3 - k - k^3 + k = 0$ ✓ TRUE

(b) Find, in terms of k, the other two roots of f.

Divide by factor

$$\begin{array}{r l}
 & x^2 - kx + 1 \\
x + k \;) & \cancel{x^3} + 0x^2 + (1-k^2)x + k \\
 & \mp \cancel{x^3} \mp kx^2 \\
 & - \cancel{kx^2} + (1-k^2)x \\
 & \pm \cancel{kx^2} \pm k^2x \\
 & 1\cancel{x} + \cancel{k} \\
 & \mp 1\cancel{x} \mp \cancel{k}
\end{array}$$

Solve quadratic

Use formula

$x^2 - kx + 1 = 0 \qquad a = 1, \; b = -k, \; c = 1$

$$x = \frac{+k \pm \sqrt{(-k)^2 - 4(1)(1)}}{2(1)} = \frac{k \pm \sqrt{k^2 - 4}}{2}$$

Express $2x^2 - 4x - 5$ in the form $a(x+h)^2 + k$ and hence,

(i) solve the equation $2x^2 - 4x - 5 = 0$

(ii) find the minimum point of this curve.

Perfect square

format

complete square

$2x^2 - 4x - 5 = 2[x^2 - 2x - 5/2] = 0$

	x	-1
x	x^2	$-x$
-1	$-x$	$+1$

$= 2[(x^2 - 2x + 2 - 1 - 5/2] = 0$

$= 2[(x-1)^2 - \frac{7}{2}] = 0$

$= 2(x-1)^2 - 7 = 0$

(i) hence solve: $2(x-1)^2 = 7$

$(x-1)^2 = 7/2$

$(x-1) = \pm\sqrt{7/2}$

$x = 1 \pm \sqrt{\frac{7}{2}}$

(ii) min. pt. $= (1, -7)$