Divisibility proofs

Example 4

Prove that for all $n \in N$, 3 is a factor of $4^n - 1$.

① $n=1$?

$4^1 - 1 = 4 - 1 = 3$ which is divisible by 3

② Assume

Assume $4^k - 1$ is divisible by 3, $k \in N$

③ Is $4^{k+1} - 1$ divisible by 3?

$4^{k+1} - 1$
$= 4^k(4) - 1$
$= 4^k(3+1) - 1$
$= 4^k(3) + (4^k - 1)$

Since 3 is a factor of both parts
$\Rightarrow$ its divisible by 3

④ State true for all $n \in N$

Since it is true for $n=k$, $n=k+1$, $n=1$
$\Rightarrow$ true for $n = 2, 3, 4 \ldots$
$\Rightarrow$ true for all $n \in N$

Exercise 7.12(B)

Prove by induction that

1. $6^n - 1$ is divisible by 5 for $n \in N$.

① $n=1$?

$6^1 - 1 = 6 - 1 = 5$, which is divisible by 5

② Assume for $n=k$

Assume $6^k - 1$ is divisible by 5, $k \in N$

③ Is $6^k - 1$ divisible by 3?

Is $6^{k+1} - 1$ divisible by 5 ?

$= 6^k(6) - 1$

$= 6^k(5+1) - 1 = (5)6^k + (6^k - 1)$

Since 5 is a factor of both parts
$\Rightarrow$ its divisible by 3

④ State true for all $n \in N$

Since it is true for $n=k$, $n=k+1$, $n=1$
$\Rightarrow$ true for $n = 2, 3, 4 \ldots$
$\Rightarrow$ true for all $n \in N$

Example 5

Prove by induction that $8^n - 7n + 6$ is divisible by 7 for all $n \in N$.

① $n=1$?	$8^1 - 7(1) + 6 = 7$, which is divisible by 7
② Assume for $n=k$	Assume $8^k + 7(k) + 6$ is divisible by 7, $k \in N$.
③ Is $8^{k+1} - 7(k+1) + 6$ divisible by 7 ?	Is $8^{k+1} - 7(k+1) + 6$ divisible by 7, $k \in N$
	$= 8^k(8) - 7k - 7 + 6$
	$= 8^k(7+1) - 7k - 7 + 6$
	$= \underbrace{(7)8^k}\ + \underbrace{(8^k - 7k + 6)}\ \underbrace{-7}$
	7 is factor of each part ⇒ its true for $n = k+1$
④ State true for all $n \in N$	Since it is true for $n=k$, $n=k+1$, $n=1$ ⇒ true for $n = 2, 3, 4 \ldots$ ⇒ true for all $n \in N$

Example 6

Show that $n(n+1)(n+2)$ is divisible by 3 for $n \in N$.

① $n=1$?	$1(1+1)(1+2) = 1(2)(3)$ which is divisible by 3
② Assume for $n=k$	Assume $k(k+1)(k+2)$ is divisible by 3, $k \in N$.
③ Is $(k+1)(k+1+1)(k+1+2)$ divisible by 3?	Is $(k+1)(k+1+1)(k+1+2)$ divisible by 3 ?
	$= (k+1)(k+2)(k+3)$
	$= k(k+1)(k+2) + 3(k+1)(k+2)$
	Since 3 is a factor of both parts ⇒ its true for $n = k+1$
④ State true for all $n \in N$	Since it is true for $n=k$, $n=k+1$, $n=1$ ⇒ true for $n = 2, 3, 4 \ldots$ ⇒ true for all $n \in N$

10. $n(n+1)(2n+1)$ is divisible by 3 for $n \in N$.

① $n=1$?

$1(1+1)(2(1)+1) = 1(2)(3)$
which is divisible by 3.

② Assume for $n=k$

Assume $k(k+1)(2k+1)$ is divisible by 3
$k \in N$

③ Is $(k+1)(k+1+1)(2(k+1)+1)$ divisible by 3 ?

Is $(k+1)((k+1)+1)(2(k+1)+1)$ divisible by 3?

$= (k+1)(k+2)(2k+2+1)$

expand $= (k+1)(k+2)(2k+1+2)$

expand $= (2k+1)(k+1)(k+2) + 2(k+1)(k+2)$

Is 3 factor of Left part? $= k(2k+1)(k+1) + 2(2k+1)(k+1) + 2(k+1)(k+2)$

$= k(2k+1)(k+1) + 2(k+1)(2k+1+k+2)$

$= k(2k+1)(k+1) + 2(k+1)(3k+3)$

$= k(2k+1)(k+1) + 3(2)(k+1)(k+1)$

3 is a factor of both terms $\Rightarrow$ $n=k+1$ is true.

④ State true for all $n \in N$

Since it is true for $n=k$, $n=k+1$, $n=1$
$\Rightarrow$ true for $n = 2, 3, 4 \ldots$
$\Rightarrow$ true for all $n \in N$

11. $n^3 - n$ is divisible by 3 for $n \in N$.

① $n=1$?

$1^3 - 1 = 0 \Rightarrow$ not true for all $n \in N$

If question said $n > 2$

① $n=2$?

$2^3 - 2 = 8 - 2 = 6 = 3(2)$ ✓

② Assume for $n=k$

Assume $k^3 - k$ is divisible $k \in N$, $k > 2$

③ Is $(k+1)^3 - (k+1)$ divisible by 3?

Is $(k+1)^3 - (k+1)$ divisible by 3 ?

$= k^3 + 3k^2 + 3k + \cancel{1} - k - \cancel{1}$

$= (k^3 - k) + 3k^2 + 3k$

$(k+1)^3 = (k+1)(k^2+2k+1)$
$= k^3 + 2k^2 + k + k^2 + 2k + 1$
$= k^3 + 3k^2 + 3k + 1$

Since 3 parts are divisible by 3
$\Rightarrow$ true for $n = k+1$

④ State true for all $n \in N$

Since it is true for $n=k$, $n=k+1$, $n=1$
$\Rightarrow$ true for $n = 2, 3, 4 \ldots$
$\Rightarrow$ true for all $n \in N$