Financial Maths

Section 5.3

Annuities (Instalment Saving)

& Pension Investments

Example 1

Catríona saves €400 every three months for five years at an effective quarterly rate of 0.9%.

(i) Represent her savings by a geometric series

(ii) Find the value of her investment at the end of the period.

(i) 5 years = 20 quarters, $0.9\% = 0.009$

$F = A(1+i)^t$

$$F = \underbrace{400(1.009)^{20}}_{T_{20}} + \underbrace{400(1.009)^{19}}_{T_{19}} + \dots + \underbrace{400(1.009)^{1}}_{T_1 = a}$$

(ii) $a = 403.6$ Ratio, $R = 1.009$ $n = 20$

$S_n = \dfrac{a(1-R^n)}{1-R}$

$$F = S_{20} = \frac{403.6(1 - 1.009^{20})}{1 - 1.009}$$

$$= €8787.81$$

Example 2

Find the sum of money, €P, that needs to be saved per month to cover the cost of a €1500 holiday in 18 months time. The interest rate on offer is 0.4% per month.

$F = P(1+i)^t$

The future value will be €1500

after 18 months of investing €P per month @ 0.4% M.E.R.

$$1500 = \underbrace{P(1\cdot004)^{18}}_{T_{18}} + \underbrace{P(1\cdot004)^{17}}_{T_{17}} + \dots + \underbrace{P(1\cdot004)^{1}}_{a = T_1}$$

$S_n = \frac{a(1-R^n)}{1-R}$

$a = 1\cdot004P$, Ratio $R = 1\cdot004$, $n = 18$

$$F = S_{18} = \frac{1\cdot004\,P\,(1 - 1\cdot004^{18})}{1 - 1\cdot004} = 1500$$

$$\Rightarrow \quad 18\cdot6998\,P = 1500$$

$$P = 1500 / 18\cdot6998$$

$$P = €80\cdot21$$

Exercise 5.3

1. Calculate the future value of 36 monthly instalments of €20.00 at an interest rate of 0.5% per month. What is the total interest earned on these savings?

$A = P(1+i)^t$

MER $= 0\cdot5\% = 0\cdot005$

geometric Series $=$

$$\underbrace{20(1\cdot005)^{36}}_{T_{36}} + \underbrace{20(1\cdot005)^{35}}_{T_{35}} + \dots + \underbrace{20(1\cdot005)^{1}}_{T_1}$$

$S_n = \frac{a(1-r^n)}{1-r}$

$T_1 = a = 20(1\cdot005)^1 = 20\cdot1$

$r = 1\cdot005$ (Common Ratio)

$n = 36$

$$S_{36} = \frac{20\cdot1\,(1 - 1\cdot005^{36})}{1 - 1\cdot005}$$

$$= €790\cdot66$$

2. Marie has saved €30.00 per month since her 18th birthday.
If her bank has guaranteed her an interest rate of 4% per annum, find

(i) the equivalent monthly rate of interest, correct to two places of decimals
(ii) the value of her savings on her 21st birthday.

(i) AER = 4% = i MER = ? = r

$(1+r)^{12} = (1+i)^1$

$(1+r)^{12} = (1+i)^1 \Rightarrow r = \sqrt[12]{1+i} - 1$

$MER = \sqrt[12]{1.04} - 1 = 0.00327$

time? terms? $\Rightarrow n=36$

(ii) 18th → 21st birthday = 3 years = 36 months

geometric series = $\underbrace{30(1.00327)^{36}}_{T_{36}} + \underbrace{30(1.00327)^{35}}_{T_{35}} + \ldots + \underbrace{30(1.00327)^1}_{T_1 = a}$

$n=36$, $a = 30(1.00327)^1 \simeq 30.1$, ratio $r = 1.00327$

$S_n = \frac{a(1-r^n)}{1-r}$

$S_{36} = \frac{30.1(1 - 1.00327^{36})}{1 - 1.00327} \simeq €1148$

Example 3

What amount of money is needed now to provide a pension of €25 000 a year for 20 years, assuming an AER of 4%?

$A = P(1+i)^t$

$\Rightarrow P = \frac{A}{(1+i)^t}$

We want to calculate a set of principals that each amount to €25 000 after amounts of time.

Geometric Series

$\text{Total} = \underbrace{\frac{25000}{(1.04)^{20}}}_{T_{20}} + \underbrace{\frac{25000}{(1.04)^{19}}}_{T_{19}} + \ldots + \underbrace{\frac{25000}{(1.04)^1}}_{T_1}$

$\times \frac{1}{1.04}$

$a = T_1$

$a = 25000/1.04 = 24\,038.46$

Ratio: $r = \frac{T_2}{T_1}$

$r = \frac{1}{1.04}$

$S_n = \frac{a(1-r^n)}{1-r}$

$S_{20} = \frac{24038.46\left(1 - \left(\frac{1}{1.04}\right)^{20}\right)}{1 - (1/1.04)}$

$= €\,339{,}758.16$

Example 4

Calculate the future value of an instalment savings plan based on saving €600 at the **start** of each year @ 4% per annum for 5 years.

(i) Calculate the present value of these payments.

(ii) Hence show that if the present value was put on deposit at the same rate for the same length of time, it would have the same future value.

Future Value?

$F = P(1+i)^t$

$$\underset{T_5}{F = 600(1\cdot04)^5} + \underset{T_4}{600(1\cdot04)^4} + \dots + \underset{a=T_1}{600(1\cdot04)^1}$$

$a = 624$, $n = 5$, Ratio $R = 1\cdot04$

$S_n = \dfrac{a(1-R^n)}{1-R}$

$$F = S_5 = \frac{624(1-1\cdot04^5)}{1-1\cdot04} = €3379\cdot79$$

(i) Present Value?

$P = \dfrac{F}{(1+i)^t}$

€600 is needed now, and in 1 year, 2 years, 3 years and 4 years time

$$P = 600 + \frac{600}{1\cdot04} + \frac{600}{1\cdot04^2} + \frac{600}{1\cdot04^3} + \frac{600}{1\cdot04^4}$$

$a = 600$, $n = 5$, Ratio $R = 1/1\cdot04$

$S_n = \dfrac{a(1-R^n)}{1-R}$

$$P = S_n = \frac{600\left(1-\left(\frac{1}{1\cdot04}\right)^5\right)}{\left(1-\frac{1}{1\cdot04}\right)} = €2777\cdot94$$

(ii) $F = P(1+i)^t$

$$F = 2777\cdot94(1\cdot04)^5 = €3379\cdot79$$

3. A special savings account offers an AER of 4% per annum. If I invest €2000 per year in this account, how much will my investment be worth in 5 years time?

$F = P(1+i)^t$

Geometric Series

$$\underset{T_5}{2000(1\cdot04)^5} + \underset{T_4}{2000(1\cdot04)^4} + \dots + \underset{T_1}{2000(1\cdot04)^1}$$

$S_n = \dfrac{a(1-R^n)}{1-R}$

$a = T_1 = 2000(1\cdot04)^1 = 2080$

Ratio: $R = 1\cdot04$

$n = 5$

$$S_5 = \frac{2080(1-1\cdot04^5)}{1-1\cdot04} =$$

$$= €11{,}265\cdot95$$

4. Show that the future value of a series of n payments of €P, earning an interest rate of i% per annum, can be written as:

$$\text{Future value} = P(1+i)\left(\frac{(1+i)^n - 1}{i}\right)$$

$F = P(1+i)^t$

assume n payments means 1 per year for n years. ?

Geometric Series

$$\underbrace{P(1+i)^n}_{T_n} + \underbrace{P(1+i)^{n-1}}_{T_{n-1}} + \dots + \underbrace{P(1+i)^1}_{T_1}$$

$S_n = \dfrac{a(1-R^n)}{1-R}$

$a = T_1 = P(1+i)$

$R = 1+i$

$n = n$

$$S_n = \frac{P(1+i)(1-(1+i)^n)}{1-(1-i)}$$

$$= P(1+i)\left(\frac{1-(1+i)^n}{\cancel{1}-\cancel{1}-i}\right)$$

multiply above and below by -1

$$= P(1+i)\left(\frac{(1+i)^n - 1}{i}\right)$$

5. Show that the present value of a series of n payments of €P, earning an interest rate of i% per annum, can be written as:

$$\text{Present value} = \frac{P}{(1+i)^n}\left(\frac{(1+i)^n - 1}{i}\right)$$

$F = P(1+i)^n$

$P = \dfrac{F}{(1+i)^n}$

Payments have been earning interest from a years down to 1 year

$$P = \underbrace{\frac{F}{(1+i)^n}}_{T_n} + \underbrace{\frac{F}{(1+i)^{n-1}}}_{T_{n-1}} + \dots + \underbrace{\frac{F}{(1+i)^1}}_{T_1 = a}$$

this is a geometric series

$a = \dfrac{F}{(1+i)}$ Ratio, $R = \dfrac{1}{(1+i)}$

Present Value =

$S_n = \dfrac{a(1-R^n)}{1-R}$

$$P = \frac{F}{1+i}\left(\frac{1-\left[\frac{1}{(1+i)}\right]^n}{1-\left(\frac{1}{1+i}\right)}\right)$$

expand denominator

$$P = F\left(\frac{1-\frac{1}{(1+i)^n}}{1+i-1}\right) = F\left(\frac{1-\frac{1}{(1+i)^n}}{i}\right)$$

multiply above and below by $(1+i)^n$

$$P = \frac{F}{(1+i)^n}\left[\frac{(1+i)^n - 1}{i}\right]$$

QED

Where F = Payments

6. Anne received a cheque in the post for €6523.33 after saving for 5 years with her bank in a scheme offering 9% per annum. If she invested €A per annum,
(i) write down a geometric series representing the value of her investment over the 5 years
(ii) find the value of A.

$F = P(1+i)^t$

Geometric Series

$$\underbrace{A(1.09)^5}_{T_5} + \underbrace{A(1.09)^4}_{T_4} + \dots + \underbrace{A(1.09)^1}_{T_1}$$

$$S_n = \frac{a(1-R^n)}{1-R}$$

$a = T_1 = A(1.09)$
Ratio: $R = 1.09$
$n = 5$
$S_5 = €6523.33$

$S_5 =$

$$6523.33 = \frac{A(1.09)(1-1.09^5)}{1-1.09}$$

$\div 6.52333$

$$6523.33 = A(6.52333)$$

$$A = €1000$$

7. Use the future value formula to find the final value if €200 is invested every month for 2 years. The interest rate is 9% per annum, compounded monthly.

change annual equivalent rate to monthly equivalent rate

9% p.a. $\Rightarrow$ MER?
$(1.09)^1 = (1+R)^{12} \Rightarrow R = \sqrt[12]{1.09} - 1$
$R = 0.0072$

Instalments?

2 years $= (12)2 = 24$ months

$F = P(1+i)^t$

Geometric Series

$$\underbrace{200(1.0072)^{24}}_{T_{24}} + \underbrace{200(1.0072)^{23}}_{T_{23}} + \dots + \underbrace{200(1.0072)^1}_{T_1}$$

$$S_n = \frac{a(1-R^n)}{1-R}$$

$a = T_1 = 200(1.0072)^1 = 201.44$
Ratio: $R = 1.0072$
$n = 24$

Future Value =

$$S_{24} = \frac{201.44(1-1.0072^{24})}{1-1.0072}$$

$$= €5256.82$$

8. George wants to make regular payments into an account that pays 8.5% per annum compound interest in order to have €10 000 after 7 years. Find the amount of each annual payment.

$F = P(1+i)^t$

Geometric Series

Principle for each month is unknown $= P$

$$P(1.085)^7 + P(1.085)^6 + \ldots + P(1.085)^1$$

$T_7 \quad T_6 \quad T_1$

$a = T_1 = P(1.085)$

Ratio $= 1.085$

$n = 7$

$S_7 = €\,10\,000$

$$S_n = \frac{a(1-R^n)}{1-R}$$

$$S_7 \Rightarrow \quad 10\,000 = \frac{P(1.085)(1-1.085^7)}{1-1.085}$$

$$10\,000 = P(9.8306)$$

$$P = 10000 / 9.8306$$

$$= €\,1017.22$$

9. Ella wants to have €5000 in 3 years time.
She invests in an annuity that pays 7.2% per annum, compounded quarterly.
How much does she need to deposit each quarter to achieve her target of €5000?

$(1+i)^1 = (1+R)^4$

7.2% p.a $\Rightarrow$ quarterly equivalent Rate? (R)

AER? $(1.072)^1 = (1+R)^4 \Rightarrow R = \sqrt[4]{1.072} - 1 = 0.0175$

number of instalments? 3 years $= 3(4) = 12$ quarters

$F = P(1+i)^t$

Let each instalment = P

Geometric Series

$$P(1.0175)^{12} + P(1.0175)^{11} + \ldots + P(1.0175)^1$$

$T_{12} \quad T_{11} \quad T_1$

$T_1 = a = P(1.0175)$

Ratio $= R = 1.0175$

$n = 12$

$S_{12} = 5000$

$$S_n = \frac{a(1-R^n)}{1-R}$$

$$S_{12} \Rightarrow \quad 5000 = \frac{P(1.0175)(1-1.0175^{12})}{1-1.0175} = P(13.456$$

$$P = 5000/13.4565 = €371.57$$

10. Prove that the present value of an annuity (instalments paid at the beginning of each period) is given by:
Future value (calculated at the end of each period) ÷ $(1 + i)^n$.

To Prove $P = \frac{F}{(1+i)^n}$

If P is invested at a rate of i for 1 period of time

1 period $\Rightarrow F = P(1+i)$

2 periods $\Rightarrow F = P(1+i)(1+i) = P(1+i)^2$

3 periods $\Rightarrow F = P(1+i)(1+i)(1+i) = P(1+i)^3$

n periods $\Rightarrow F = P(1+i)^n$

$$\Rightarrow P = \frac{F}{(1+i)^n}$$

11. Show how the present value of an annuity involving depositing €3000 per year in an account for 6 years can be written as a geometric series, given that the interest rate is 8% per annum.
(i) Calculate the present value.
(ii) Calculate the future value of the annuity.
(iii) If the present value of the annuity in (i) was put on deposit as a single investment at 8% per annum, show that it will amount to the same future value of the annuity after 6 years.

(i)

The first instalment is €3000 this is needed immediately and won't be earning interest

$P = \frac{F}{(1+i)^t}$

Geometric Series

$$\underset{T_6}{\frac{3000}{(1\cdot08)^5}} + \underset{T_5}{\frac{3000}{(1\cdot08)^4}} + \dots \underset{T_2}{\frac{3000}{(1\cdot08)^1}} + \underset{T_1}{3000}$$

$S_n = \frac{a(1-R^n)}{1-R}$

$T_1 = a = 3000$
$R = 1/1{\cdot}08$
$n = 6$

Present Value =

$$S_6 = \frac{3000\left(1-(1/1{\cdot}08)^6\right)}{1-(1/1{\cdot}08)}$$

$= €14{,}978{\cdot}13$ ✓

11. Show how the present value of an annuity involving depositing €3000 per year in an account for 6 years can be written as a geometric series, given that the interest rate is 8% per annum.
(i) Calculate the present value.
(ii) Calculate the future value of the annuity.
(iii) If the present value of the annuity in (i) was put on deposit as a single investment at 8% per annum, show that it will amount to the same future value of the annuity after 6 years.

(ii)

$F = P(1+i)^t$

Geometric Series

$$\underbrace{3000(1.08)^6}_{T_6} + \underbrace{3000(1.08)^5}_{T_5} + \ldots + \underbrace{3000(1.08)^1}_{T_1}$$

$S_n = \frac{a(1-R^n)}{1-R}$

$a = T_1 = 3000(1.08) = 3240$

Ratio, $R = 1.08$

$n = 6$

$$S_6 = \frac{3240(1-1.08^6)}{1-1.08} = €\,23{,}768.41 \checkmark$$

11. Show how the present value of an annuity involving depositing €3000 per year in an account for 6 years can be written as a geometric series, given that the interest rate is 8% per annum.
(i) Calculate the present value.
(ii) Calculate the future value of the annuity.
(iii) If the present value of the annuity in (i) was put on deposit as a single investment at 8% per annum, show that it will amount to the same future value of the annuity after 6 years.

(iii)

$F = P(1+i)^t$

$P = €14{,}978.13$

$i = 8\%$

$t = 6$ years

$F = 14978.13(1.08)^6$

$= €\,23{,}768.20 \checkmark$

Summary

The future value of n payments of €P at i%

$$\text{Future value} = P(1+i)\left(\frac{1-(1+i)^n}{1-(1+i)}\right)$$

$$= P(1+i)\left(\frac{(1+i)^n - 1}{i}\right)$$

The present value (cost) of n payments of €P at i%

$$\text{Present value} = \left(\frac{P}{1+i}\right)\left[\frac{1-\left(\frac{1}{1+i}\right)^n}{1-\left(\frac{1}{1+i}\right)}\right]$$

$$= \frac{P}{(1+i)^n}\left(\frac{(1+i)^n - 1}{i}\right)$$