Geometric series

$r = \frac{T_2}{T_1}$

$T_n = ar^{n-1}$

$S_n = \frac{a(1-r^n)}{1-r}$

Example 1

Find T_5 and S_5 of each of the following:

(i) $1 + 3 + 9 + \ldots$

$a = 1 \qquad r = 3/1 = 3$

$T_5 = 1(3)^4 = 81$

$S_5 = \frac{1(1-3^5)}{1-3}$

$= 121$

ii) $1 + \frac{1}{4} + \frac{1}{16} + \ldots$

$a = 1 \qquad r = \frac{(1/4)}{1} = 1/4$

$T_5 = 1\left(\frac{1}{4}\right)^4 = \frac{1}{256}$

$S_5 = \frac{1\left(1-\left(\frac{1}{4}\right)^5\right)}{1-\left(\frac{1}{4}\right)}$

$= \frac{341}{256}$

Geometric series

$T_n = ar^{n-1}$

divide

$S_n = \frac{a(1-r^n)}{1-r}$

Example 2

In a geometric series, $T_3 = 32$ and $T_6 = 4$; find a and r and hence find S_8, the sum of the first eight terms.

$T_3 = 32 \qquad T_6 = 4$

$32 = ar^2 \qquad 4 = ar^5$

$\frac{\cancel{a}r^5}{\cancel{a}r^2} = \frac{4}{32} \Rightarrow r^3 = \frac{1}{8}$

$\Rightarrow \sqrt[3]{\frac{1}{8}} = r = \frac{1}{2}$

$\Rightarrow a = 32/\left(\frac{1}{2}\right)^2 = 32(4) = 128 = a$

$S_8 = \frac{128\left(1-\left(\frac{1}{2}\right)^8\right)}{1-\frac{1}{2}} = 255$

P.160

Ex. 4.5

Q4

Series: $32 + 16 + 8 + \ldots$ $S_{10} = ?$

$a = 32$ $r = \frac{16}{32} = \frac{1}{2}$

$S_n = \frac{a(1-r^n)}{1-r}$

$$S_{10} = \frac{32\left(1 - \left(\frac{1}{2}\right)^{10}\right)}{1 - \frac{1}{2}}$$

$$\simeq 63.94$$

Q5

Series: $4 - 12 + 36 - 108 + \ldots$

$S_6 = ?$

$a = 4$ $r = \frac{-12}{4} = -3$

$S_n = \frac{a(1-r^n)}{1-r}$

$$S_6 = \frac{4\left(1 - (-3)^6\right)}{1 - (-3)}$$

$$= -728$$

Q6

Series: $729 - 243 + 81 - \dots - \frac{1}{3}$

Find the number of terms?

$n = ?$

$a = 729$, $r = \frac{-243}{729} = -\frac{1}{3}$, $T_n = -\frac{1}{3}$

$T_n = ar^{n-1}$

$\Rightarrow -\frac{1}{3} = 729\left(-\frac{1}{3}\right)^{n-1}$

divide by 729

$\frac{1}{(-3)(729)} = \frac{1}{(-3)^{n-1}}$

$(3)(729) = 2187$

$\log_3 2187 = 7$

$\Rightarrow \frac{1}{(-3)^7} = \frac{1}{(-3)^{n-1}}$ $\Rightarrow n-1 = 7$

$\Rightarrow n = 8$

$S_n = \frac{a(1-r^n)}{1-r}$

$S_8 = \frac{729\left(1-\left(-\frac{1}{3}\right)^8\right)}{1-\left(-\frac{1}{3}\right)} = 546\frac{2}{3}$

HW ex 4.5 Q4 - 7

Q7

$$\sum_{r=1}^{6} 4^r$$

Sum terms from input r=1 to r=6

T_n

$T_1 = 4^1 = 4$

$T_2 = 4^2 = 16$ $\times 4$

$T_3 = 4^3 = 64$

$a = 4$

$r = 4$

$S_n = \frac{a(1-r^n)}{1-r}$

$S_6 = \frac{4(1-4^6)}{1-4} = 5460$

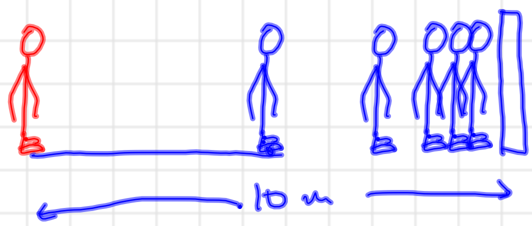
For a geometric series with $|r| < 1$,

$$\lim_{n \to \infty} S_n = \frac{a}{1-r}.$$

$|r| < 1$

The idea of a sum of infinite terms having a limit.

If I walk towards a wall that is 10 m away and every second I cover half the distance between me and the wall. I will never reach the wall. The sum of all the distances I cover will add up to slightly less than 10 m!



For a geometric series with $|r| < 1$,

$$\lim_{n \to \infty} S_n = \frac{a}{1-r}.$$

Example 3

Find the sum to infinity of the geometric series $16 + 12 + 9 + \ldots..$

$a = 16$ $\quad r = \frac{12}{16} = \frac{3}{4}$

$$S_\infty = \frac{a}{1-r} = \frac{16}{1-\frac{3}{4}} = \frac{16}{\frac{1}{4}} = 64$$

19. The value of a sum of money on deposit at 3% per annum compound interest is given by $A = €4000\,(1.03)^t$ where t is the number of years of the investment. Find

(i) the amount of money on deposit

(ii) the value of the investment at the end of each of the first four years

(iii) the value of the investment at the end of the 10th year

(iv) the number of years, correct to the nearest year, needed for the investment to double in value.

(i) on deposit $\Rightarrow t = 0$ years

$\Rightarrow A_0 = 4000\,(1.03)^0 = €\,4000$

(ii) 1 year $\Rightarrow A_1 = 4000(1.03)^1 = €\,4120$

2 year $\Rightarrow A_2 = 4000\,(1.03)^2 = €\,4243.60$

3 year $\Rightarrow A_3 = 4000\,(1.03)^3 = €\,4370.91$

4 years $\Rightarrow A_4 = 4000\,(1.03)^4 = €\,4502.04$

(iii) $A_{10} = 4000\,(1.03)^{10} = €\,5375.67$

$A_0 = €4000$

Double $A_0 = €8000$

$t = ?$

$A = 4000\,(1.03)^t$

$\Rightarrow 8000 = 4000\,(1.03)^t$

$\Rightarrow 2 = 1.03^t$

$t = \log_{1.03} 2 = 23.45 \simeq 23$ years (n.w.y)

Recurring decimals

Recurring decimals can be expressed as a sum to infinity of a geometric sequence, where the common ratio $r < 1$.

For example, $0.\dot{3} = 0.3333\ldots.. = \dfrac{3}{10} + \dfrac{3}{10^2} + \dfrac{3}{10^3} + \dfrac{3}{10^4} + \ldots..$ (each term $\times \frac{1}{10}$)

where $a = 0.3$ and $r = \dfrac{1}{10}$.

Similarly,

$$0.2\dot{3}\dot{5} = 0.2353535\ldots.. = 0.2 + [0.035 + 0.00035 + \ldots..]$$

$$= 0.2 + \frac{35}{1000} + \frac{35}{100000} + \ldots..$$

$$= 0.2 + \text{an infinite geometric series}$$

where $a = \dfrac{35}{1000}$ and $r = \dfrac{1}{100}$.

10. Write each of the following recurring decimals as an infinite geometric series. Hence express each as a decimal in the form $\frac{a}{b}$, $a, b \in N$.

(i) $0.\dot{7}$ (ii) $0.\dot{3}\dot{5}$ (iii) $0.2\dot{3}$ (iv) $0.\dot{3}7\dot{0}$ (v) $0.1\dot{6}\dot{2}$ (vi) $0.3\dot{2}\dot{1}$

Ex.4.5

(i) $0.\dot{7} = 0.7777\ldots = \frac{7}{10} + \frac{7}{100} + \frac{7}{1000} + \ldots$

$S_\infty = \frac{a}{1-r}$

$a = \frac{7}{10}$ $r = \frac{1}{10}$

$S_\infty = \frac{7/10}{1-1/10} = \frac{7/10}{9/10} = \frac{7}{9}$

(ii) $0.\dot{3}\dot{5} = 0.353535\ldots = \frac{35}{100} + \frac{35}{10000} + \frac{35}{1000000} + \ldots$

$a = 35/100$ $r = 1/100$

$S_\infty = \frac{a}{1-r}$

$S_\infty = \frac{(35/100)}{(1-1/100)} = \frac{(35/100)}{(99/100)} = \frac{35}{99}$

11. Find S_n, the sum to n terms, of $1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \ldots\ldots \left(\frac{1}{2}\right)^{n-1}$ and hence find S_∞, the sum to infinity of the series.
Find the least value of n such that $S_\infty - S_n < 0.001$.

$S_n = \frac{a(1-r^n)}{1-r}$

$a = 1$ $r = \frac{1}{2}$ $n = n$

$S_n = \frac{1(1-(\frac{1}{2})^n)}{1-1/2} = \frac{1(1-(\frac{1}{2})^n)}{1/2} = 2(1-(\frac{1}{2})^n)$

$S_\infty = \frac{a}{1-r}$

$S_\infty = \frac{1}{(1-\frac{1}{2})} = \frac{1}{(\frac{1}{2})} = 2$

$S_\infty - S_n < 0.001$

$\Rightarrow 2 - 2(1-(\frac{1}{2})^n) < 0.001$

Subtract 2

$-2(1-(\frac{1}{2})^n) < -1.999$

divide by -2 & change inequality

$1-(\frac{1}{2})^n > 0.9995$

Subtract 1

$-\frac{1}{2}^n > -0.0005$

change signs & inequality

$(\frac{1}{2})^n < 0.0005$

$\log_{\frac{1}{2}} 0.0005 = 10.965$

$\Rightarrow (\frac{1}{2})^{11} < 0.0005 \Rightarrow n = 11$