Section 2.1 Quadratic equations

2. Use the quadratic formula to solve each of the following, giving your answers correct to one place of decimals:

(a) (i) $x^2 - 2x - 2 = 0$

Remember...
If $ax^2+bx+c=0$

$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$a = 1$
$b = -2$
$c = -2$

$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-2)}}{2(1)}$

$= \frac{2 \pm \sqrt{4+8}}{2} = \frac{2 \pm \sqrt{12}}{2} = \frac{2 \pm 2\sqrt{3}}{2} = 1 \pm \sqrt{3}$

$x_1 = 1 + \sqrt{3} = 2{\cdot}7$

$x_2 = 1 - \sqrt{3} = -0{\cdot}7$

3. Use the quadratic formula to solve each of the following, leaving your answers in surd form:

(a) (i) $3x^2 + 4x - 5 = 0$

Remember...
If $ax^2+bx+c=0$

$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$a = 3$
$b = 4$
$c = -5$

$x = \frac{-4 \pm \sqrt{(4)^2 - 4(3)(-5)}}{2(3)}$

$= \frac{-4 \pm 2\sqrt{19}}{6} = \frac{-2 \pm \sqrt{19}}{3}$

$x_1 = \frac{-2 + \sqrt{19}}{3}$

$x_2 = \frac{-2 - \sqrt{19}}{3}$

4. Solve the following equations:

(a) (i) $\dfrac{x+7}{3} + \dfrac{2}{x} = 4$

multiply by LCM = 3x

$$x(x+7) + 3(2) = 4(3x)$$

$$x^2 + 7x + 6 = 12x$$

$$x^2 - 5x + 6 = 0$$

$$(x-2)(x-3) = 0$$

$$x = 2 \mid x = 3$$

5. By finding a suitable substitution, solve each of the following:

(c) $\left(y + \dfrac{4}{y}\right)^2 - 9\left(y + \dfrac{4}{y}\right) + 20 = 0$

let $x = y + \frac{4}{y}$ ⇒ Rewrite as

$$x^2 - 9x + 20 = 0$$

$$(x-4)(x-5) = 0$$

$$x = 4 \mid x = 5$$

Sub back in

$4 = y + \frac{4}{y}$	$5 = y + \frac{4}{y}$
$4y = y^2 + 4$	$5y = y^2 + 4$
$y^2 - 4y + 4 = 0$	$y^2 - 5y + 4 = 0$
$(y-2)(y-2) = 0$	$(y-4)(y-1) = 0$
$y = 2 \mid y = 2$	$y = 4 \mid y = 1$

10. The graphs of the functions

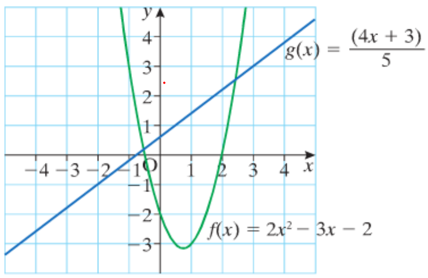
$f(x) = 2x^2 - 3x - 2$ and $g(x) = \dfrac{4x + 3}{5}$

are drawn as shown. Using the graphs, estimate the solutions of the following equations

(a) $f(x) = 0$ where green curve cuts x-axis

(b) $g(x) = 0$ where blue line cuts x-axis

(c) $f(x) = g(x)$. Intersection of blue line and green curve



(a) $x = -0.5$ and $x = 2$

(b) $x = -0.7$

(c) $x = -0.6$ and $x = 2.4$

Section 2.2 Nature of quadratic roots

9. Prove that the equation $(k - 2)x^2 + 2x - k = 0$ has real roots, whatever the value of k.

1. If $(b^2 - 4ac) > 0 \longrightarrow$ two different (distinct) real roots
2. If $(b^2 - 4ac) = 0 \longrightarrow$ two equal real roots
3. If $(b^2 - 4ac) < 0 \longrightarrow$ two imaginary roots
4. If $(b^2 - 4ac)$ is a perfect square $\longrightarrow$ rational roots

Remember...

The square of a real number ≥ 0

$x^2 \geq 0,\ x \in R$

$a = k-2$

$b = 2$

$c = -k$

$b^2 - 4ac = (2)^2 - 4(k-2)(-k) = 4 + 4k^2 - 8k$

$= 4(k^2 - 2k + 1)$

$= 4(k-1)(k-1) = 4(k-1)^2 \geq 0$

$\Rightarrow$ it has real roots

3. Find the discriminant of each of the following equations and state if the roots are (a) real and different (b) real and equal (c) imaginary.

(i) $2x^2 + x + 5 = 0$ (ii) $-2x^2 + 3x + 1 = 0$ (iii) $3x^2 + 2x - 1 = 0$

(iv) $-3 + 2x - x^2 = 0$ (v) $x^2 + 8x + 16 = 0$ (vi) $25 - 10x + x^2 = 0$

Remember...

Discriminant = Δ

$\Delta = b^2 - 4ac$

(i) $a = 2$, $b = 1$, $c = 5$

$\Delta = (1)^2 - 4(2)(5)$

$= 1 - 40$

$= -39 < 0$

imaginary roots

(ii) $a = -2$ $b = 3$, $c = 1$

$\Delta = (3)^2 - 4(-2)(1)$

$= 9 + 8$

$= 17 > 0$

real and different

(iii) $a = 3$, $b = 2$, $c = -1$

$\Delta = (2)^2 - 4(3)(-1)$

$= 4 + 12$

$= 16 > 0$

real and different

(iv) $a = -1$, $b = 2$, $c = -3$

$\Delta = (2)^2 - 4(-1)(-3)$

$= 4 - 12$

$= -8 < 0$

imaginary roots

(v) $a = 1$, $b = 8$, $c = 16$

$\Delta = (8)^2 - 4(1)(16)$

$= 64 - 64$

$= 0$

real and equal

(vi) $a = 1$, $b = -10$, $c = 25$

$\Delta = (-10)^2 - 4(1)(25)$

$= 100 - 100$

$= 0$

real and equal

10. Find the value of k for which the equation $(k-2)x^2 + x(2k+1) + k = 0$ has equal roots.

Remember...
Discriminant = Δ
$\Delta = b^2 - 4ac$

Remember...
When roots are equal
$\Delta = 0$

$a = k-2$
$b = 2k+1$
$c = k$

$$\Rightarrow \quad (2k+1)^2 - 4(k-2)(k) = 0$$
$$\cancel{4k^2} - 4k + 1 - \cancel{4k^2} - 8k = 0$$
$$1 = 12k$$
$$k = \frac{1}{12}$$

13. Show that the equation $x^2 - 2px + 3p^2 + q^2 = 0$ cannot have real roots for $p, q \in R$.

Remember...
Discriminant = Δ
$\Delta = b^2 - 4ac$

Remember...
When roots are real
$\Delta \leq 0$

$a = 1 \qquad b = -2p \qquad c = 3p^2 + q^2$

$$\Delta = (-2p)^2 - 4(1)(3p^2 + q^2)$$
$$= 4p^2 - 12p^2 - 4q^2$$
$$= -8p^2 - 4q^2 \leq 0$$

imaginary roots

Section 2.3 Solving quadratic and linear equations

Solve: **12.** $x^2 + y^2 + 2x - 4y + 3 = 0$

$x - y + 3 = 0$ Step 1: Rewrite linear $\Rightarrow$ $x = y - 3$

Step 2: Sub into quadratic & solve

$$(y-3)^2 + y^2 + 2(y-3) - 4y + 3 = 0$$

$$y^2 - 6y + 9 + y^2 + 2y - 6 - 4y + 3 = 0$$

$$2y^2 - 8y + 6 = 0$$

$$y - 4y + 3 = 0$$

$$(y-1)(y-3) = 0$$

$$y = 1 \quad | \quad y = 3$$

Step 3: Sub values into linear to find points

$$x = y - 3$$

$$x = 1 - 3 = -2$$

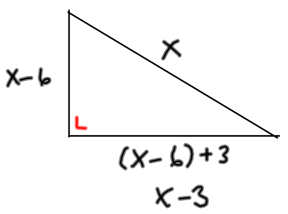
$$(-2, 1)$$

$$x = 3 - 3 = 0$$

$$(0, 3)$$

Section 2.4 Quadratic and linear equations in context

10. The hypotenuse of a right-angled triangle is 6 cm longer than the shortest side. The third side is 3 cm longer than the shortest side. Find the length of the shortest side.



Remember...

Pythagoras

$a^2 = b^2 + c^2$

$$x^2 = (x-6)^2 + (x-3)^2$$

$$x^2 = x^2 - 12x + 36 + x^2 - 6x + 9$$

$$x^2 - 18x + 45 = 0$$

$$(x-3)(x-15) = 0$$

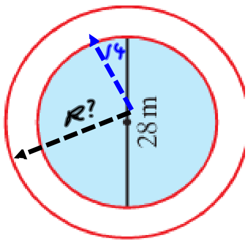
$$x = 15 \quad | \quad x = 3$$

this answer implies one side is 0 so it doesn't make sense.

$\Rightarrow$ hypotenuse $= 15$ cm

Shortest side $= 15 - 6 = 9$ cm

13. A circular swimming pool with a diameter of 28 metres has a wooden deck around its edge.
If the deck has an area of $60\pi\,\text{m}^2$, find the width of the deck.



Pool Area $= \pi(14)^2 = 196\pi$

R = Radius of pool + deck?

Pool + Deck Area $= 196\pi + 60\pi = 256\pi$

$$256\cancel{\pi} = \pi^2 R$$

$$R = \sqrt{256} = 16$$

Withof deck $= 16 - 14 = 2$ m

Remember...

Area of a disc = πr^2

Section 2.5 Forming quadratic equations from their roots

1. State (i) the sum and (ii) the product of the roots of each of the following quadratic equations.

(a) $x^2 + 9x + 4 = 0$

(i) $r_1 + r_2 = -9$

(ii) $r_1 r_2 = 4$

(b) $x^2 - 2x - 5 = 0$

(i) $r_1 + r_2 = 2$

(ii) $r_1 r_2 = -5$

Remember...

$$X^2 - \left(\text{Sum of Roots}\right)x + \left(\text{Product of Roots}\right) = 0$$

3. Find the quadratic equations that have the following pairs of roots (r_1, r_2).

(iv) $(\sqrt{5}, 4)$

$$x^2 - (4+\sqrt{5})x + 4\sqrt{5} = 0$$

Remember...

$$x^2 - \left(\begin{smallmatrix}\text{Sum of}\\ \text{Roots}\end{smallmatrix}\right)x + \left(\begin{smallmatrix}\text{Product}\\ \text{of Roots}\end{smallmatrix}\right) = 0$$

(viii) $\left(\frac{5}{2}, \frac{3}{5}\right)$

$$x^2 - \left(\frac{5}{2} + \frac{3}{5}\right)x + \left(\frac{5}{2}\right)\left(\frac{3}{5}\right) = 0$$

$$x^2 - \left(\frac{25+6}{10}\right)x + \frac{15}{10} = 0$$

$$10x^2 - 31x + 15 = 0$$

Section 2.6 Max and Min of Quadratic graphs

3. Write each of the following in the form $(x - p)^2 + q = 0$.

(i) $x^2 + 4x - 6 = 0$

	x	2
x	x^2	$2x$
2	$2x$	4

$$\underbrace{x^2 + 4x + 4}\ \underbrace{-6 - 4} = 0$$

$$(x+2)^2 - 10 = 0$$

9. If $f(x) = x^2 + 4x + 7$, find
(i) the smallest possible value of $f(x)$
(ii) the value of x at which this smallest value occurs
(iii) the greatest possible value of $\dfrac{1}{(x^2 + 4x + 7)}$.

min. pt.?

	x	2
x	x^2	$2x$
2	$2x$	4

$$f(x) = \underbrace{x^2 + 4x + 4}\ \underbrace{+7 - 4}$$
$$= (x+2)^2 + 3$$

min. pt. $(-2, 3)$

(i) 3
(ii) -2

(iii) the greatest possible value of $\dfrac{1}{(x^2 + 4x + 7)}$.

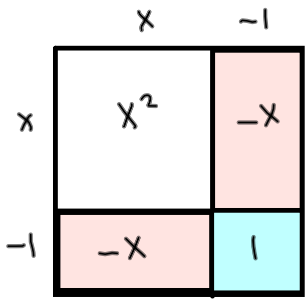
the value of this fraction is greatest when the denominator has its minimum value

this occurs at $x = -2$ so sub in

$$\frac{1}{(-2)^2 + 4(-2) + 7} = \frac{1}{4 - 8 + 7} = \frac{1}{3}$$

4. The graph of $y = a(x - p)^2 + q$ has a minimum point (p, q).
By completing the square, find the minimum point of each of the following quadratic equations:

(ii) $3x^2 - 6x - 1 = 0$



$y = 3[x^2 - 2x - \frac{1}{3}]$

$= 3[\underline{x^2 - 2x + 1} \ \underline{-\frac{1}{3} - 1}]$

$= 3[(x-1)^2 - \frac{4}{3}]$

$= 3(x-1)^2 - 4$

min. pt. $= (1, -4)$

Section 2.7 Surds

2. Express each of the following in its simplest form:

(i) $2\sqrt{2} + 6\sqrt{2} - 3\sqrt{2}$

$= 8\sqrt{2} - 3\sqrt{2}$

$= 5\sqrt{2}$

(ii) $2\sqrt{2} + \sqrt{18}$

$= 2\sqrt{2} + \sqrt{9(2)}$

$= 2\sqrt{2} + 3\sqrt{2}$

$= 5\sqrt{2}$

5. By rationalising the denominator, express each of the following in its simplest form.

(ii) $\dfrac{12}{3-\sqrt{2}}$

Remember...

When dividing by a Compound Surd multiply above and below by its conjugate

Remember...

Conjugate of

(a + √b) is (a - √b)

$$= \frac{12(3+\sqrt{2})}{(3-\sqrt{2})(3+\sqrt{2})} = \frac{12(3+\sqrt{2})}{9-2} = \frac{12(3+\sqrt{2})}{7}$$

notice difference of 2 squares

Section 2.8 Algebraic surd equations

4. Show that $\dfrac{-1+\sqrt{3}}{1+\sqrt{3}} = 2-\sqrt{3}$. easy way is to multiply by LCD

if its true then

$$\begin{aligned} -1+\sqrt{3} &= (2-\sqrt{3})(1+\sqrt{3}) \\ &= 2-2\sqrt{3}-\sqrt{3}-3 \\ &= -1\sqrt{3} \quad \checkmark \end{aligned}$$

7. Solve the following equations and check your solutions in each case:

- If there is **only one surd**, isolate it on one side and then square both sides and solve.
- If there are **two surds**, move one to each side of the equation. Square both sides and isolate any remaining surds. Square both sides again to remove any remaining surd.
- Solve the resulting equation.
- Check your answers.

(iv) $\sqrt{3x-5} = x-1$

SQUARE BOTH SIDES

$3x-5 = x^2-2x+1$

$x^2-5x+6=0$

$(x-3)(x-2)=0$

$x=3 \mid x=2$

Check: $\sqrt{3(3)-5} \stackrel{?}{=} 3-1$

$\sqrt{9-5} \stackrel{?}{=} 2$

$\sqrt{4} \stackrel{?}{=} 2$ ✓

$\sqrt{3(2)-5} \stackrel{?}{=} 2-1$

$\sqrt{6-5} \stackrel{?}{=} 1$

$\sqrt{1} \stackrel{?}{=} 1$ ✓ both answers are valid

8. Solve each of these equations and check each solution:

(iv) $\sqrt{3x-2} = \sqrt{x-2} + 2$

Remember...

$(a+b)^2 = a^2+2ab+b^2$

SQUARE BOTH SIDES

$3x-2 = (x-2) + 4\sqrt{x-2} + 4$

$3x-2 = x + 4\sqrt{x-2} + 2$

$2x-4 = 4\sqrt{x-2}$

SQUARE BOTH SIDES

$4x^2-16x+16 = 16(x-2)$

$4x^2-16x+16 = 16x-32$

$4x^2-32x+48=0$

$x^2-8x+12=0$

$(x-6)(x-2)=0$

$x=6 \mid x=2$

Section 2.9 The factor theorem

6. Show that $(2x - 1)$ is a factor of $2x^3 + 7x^2 + 2x - 3$.

> Remember...
>
> If (x - a) is a factor
>
> f(a) = 0

if $(2x-1)$ is a factor

then $f\left(\frac{1}{2}\right) = 0$

check: $f\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right)^3 + 7\left(\frac{1}{2}\right)^2 + 2\left(\frac{1}{2}\right) - 3$

$= \frac{2}{8} + \frac{7}{4} + 1 - 3$

$= \frac{1}{4} + \frac{7}{4} - 2 = 0$ ✓

15. Factorise fully $x^3 - x^2 - 14x + 24$.

Hence solve the equation $x^3 - x^2 - 14x + 24 = 0$.

$f(1) = (1)^3 - (1)^2 - 14(1) + 24 > 0$

$f(2) = (2)^3 - (2)^2 - 14(2) + 24 = 0$

So $(x-2)$ is a factor

> Remember...
>
> If f(a) = 0
>
> then (x - a) is a factor

DIVIDE

$$\begin{array}{r|l} & x^2 + x - 12 \\ \hline x-2 & x^3 - x^2 - 14x + 24 \\ & \mp x^3 \pm 2x^2 \\ \hline & \quad x^2 - 14x \\ & \quad \mp x^2 \pm 2x \\ \hline & \qquad -12x + 24 \\ & \qquad \pm 12x \mp 24 \\ \hline & \qquad\qquad 0 \end{array}$$

FACTORISE

$x^2 + x - 12$

$(x-3)(x+4)$

FACTORS ARE:

$(x-2)(x-3)(x+4)$

SOLUTIONS ARE

$x = 2, \quad x = 3, \quad x = -4$

20. If $(x+2)$ and $(x-3)$ are both factors of $2x^3+ax^2-17x+b$, find the values of a and b.

Hence find the third factor.

Remember...

If (x - a) is a factor

f(a) = 0

$f(-2) = 2(-2)^3 + a(-2)^2 - 17(-2) + b = 0$

$-16 + 4a + 34 + b = 0$

$4a + b = -18$

$f(3) = 2(3)^3 + a(3)^2 - 17(3) + b = 0$

$54 + 9a - 51 + b = 0$

$9a + b = -3$

$9a + b = -3$

$-4a - b = 18$

$5a = 15$

$a = 3$

$4(3) + b = -18$

$12 + b = -18$

$b = -30$

$2x^3 + 3x^2 - 17x - 30$

multiply given factors : $(x+2)(x-3) = x^2 - 3x + 2x - 6 = x^2 - x - 6$

divide

$$\begin{array}{r|l} & 2x + 5 \\ x^2 - x - 6 & 2x^3 + 3x^2 - 17x - 30 \\ & \mp 2x^3 \pm 2x^2 \pm 12x \\ \hline & 5x^2 - 5x - 30 \\ & \mp 5x^2 \pm 5x \pm 30 \\ \hline & 0 \end{array}$$

other factor is

$(2x+5)$

Section 2.10 Graphs of cubic polynomials

7. Given* $f(x) = (x+2)(x-1)(x-3)$, find the values of $f(0)$, $f(\frac{1}{2})$and $f(2)$.
Hence draw a rough sketch of the curve.

$$f(0) = (0+2)(0-1)(0-3) = (2)(-1)(-3) = 6$$

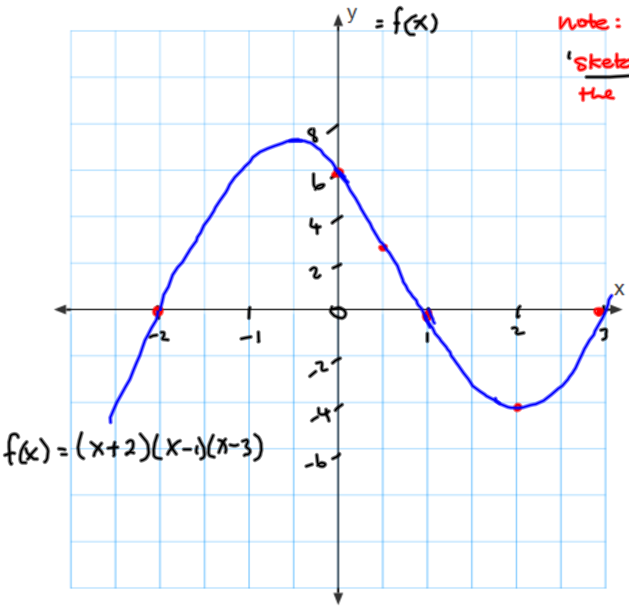
$$f(\tfrac{1}{2}) = (\tfrac{1}{2}+2)(\tfrac{1}{2}-1)(\tfrac{1}{2}-3) = \left(\tfrac{5}{2}\right)\left(-\tfrac{1}{2}\right)\left(-\tfrac{5}{2}\right) = \tfrac{25}{8} = 3\tfrac{1}{8}^{*}$$

* Answer in the book is incorrect

$$f(2) = (2+2)(2-1)(2-3) = (4)(1)(-1) = -4$$

* from factors we know (-2, 0), (1, 0) and (3, 0) are points on the curve

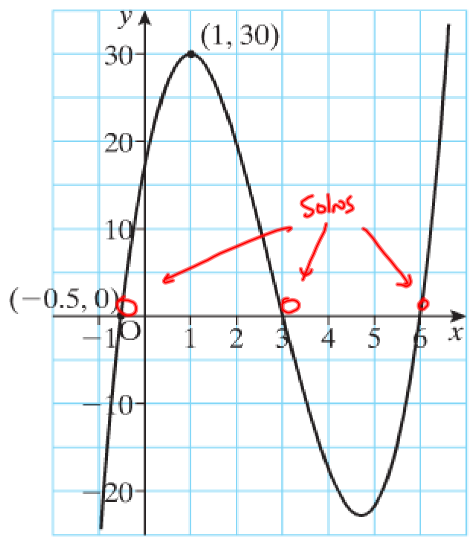
x	f(x)
-2	0
0	6
$\frac{1}{2}$	$3\frac{1}{8}$
1	0
2	-4
3	0



note: this is a 'sketch' based on the given points

11. Find a cubic expression for each of the following curves.

(i)



$ax^3+bx^2+cx+d=0$

$(x-3)(x-6)(x+\frac{1}{2})=0$

Is $f(x)=(x-3)(x-6)(x+\frac{1}{2})$?

Check: $f(1)=(1-3)(1-6)(1+\frac{1}{2})$

$=(-2)(-5)(1.5)=15\neq 30$

Scale factor of 2 needed.

$\Rightarrow f(x)=2(x-3)(x-6)(x+\frac{1}{2})$

$=(x-3)(x-6)(2x+1)$

Expand... ⋮

$f(x)=2x^3-17x^2+27x+18$

11. Find a cubic expression for each of the following curves.

$(x+4)(x+\frac{1}{2})(x-2\frac{1}{2})=0$

Is $f(x)=(x+4)(x+\frac{1}{2})(x-2\frac{1}{2})$

check $f(0)=(0+4)(0+\frac{1}{2})(0-2\frac{1}{2})$

$=(4)(\frac{1}{2})(-\frac{5}{2})=-5\neq 20$

Scale factor of -4 needed

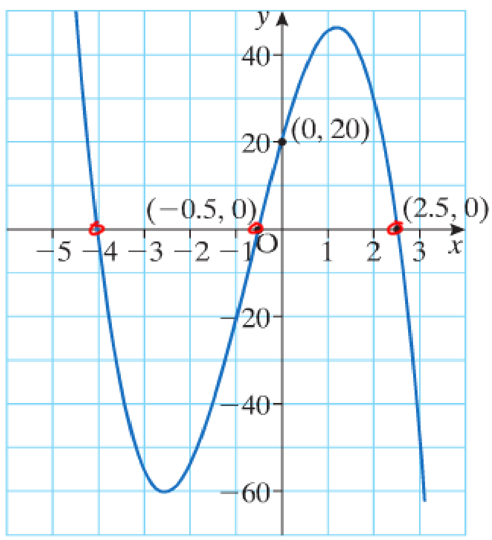
$f(x)=-4(x+4)(x+\frac{1}{2})(x-\frac{5}{2})$

$=(x+4)(2x+1)(5-x)$

expand ... ⋮

$f(x)=-4x^3-8x^2+37x+20$

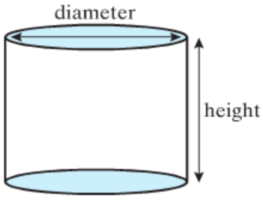
(ii)



15. The volume of a cylinder is given by $V = \pi r^2 h$, where r is the radius and h is the height. Given that the diameter is equal to the height, show that the volume can be written as

$$V = ah^3.$$

Taking $\pi = 3.14$, find the value of a correct to two places of decimals.
Using this function, calculate the volume of a cylinder with a diameter of 11 cm.
Find the diameter of a cylinder whose volume is 215.58 cm^3, correct to one place of decimals.



$2r = h$

$r = \frac{h}{2}$

$V = \pi\left(\frac{h}{2}\right)^2 h = \frac{\pi}{4} h^3$

$\Rightarrow a = \frac{\pi}{4} = 0.79$

$V = 0.79\, d^3$

DIAMETER = 11

$V = 0.79(11)^3 = 1051.49 \text{ cm}^3$

$V = 0.79\, d^3$

$215.58 = 0.79\, d^3$

$d^3 = 215.58/0.79 =$

$d = \sqrt[3]{215.58/0.79} = 6.5 \text{ cm}$