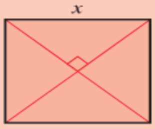
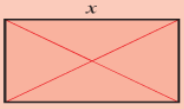
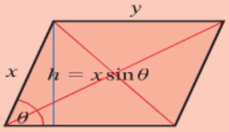
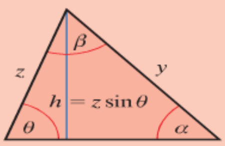
Area and Volume

Section 6.1

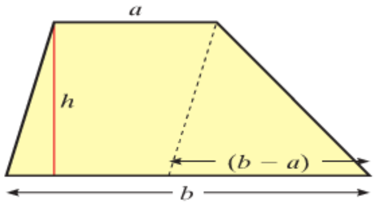
Key words

polygon **area** **perimeter** **diagonal** **trapezium** **cyclic quadrilateral** **arc** **sector** **radian** **quadrilateral**

Shape	**Diagram**	**Properties**
Square		› all sides have the same length › all angles are 90° › perimeter $= 4x$ › area $= x^2$ › diagonal $= \sqrt{2}x$ › diagonals perpendicularly bisect each other
Rectangle		› opposite sides have the same length › all angles are 90° › perimeter $= 2(x + y)$ › area $= xy$ › diagonal $= \sqrt{x^2 + y^2}$ › diagonals have the same length › diagonals bisect each other
Parallelogram		› opposite sides have the same length › opposite angles are equal › perimeter $= 2(x + y)$ › area $= yh = yx\sin\theta$ › diagonals bisect each other
Triangle		› perimeter $= x + y + z$ › area $= \frac{1}{2}x.h = \frac{1}{2}x.z.\sin\theta$ › $\frac{y}{\sin\theta} = \frac{z}{\sin\alpha}$ › types include isosceles, equilateral, scalene, right-angled › $\alpha + \beta + \theta = 180°$ › special right-angled triangles with sides • $3, 4, \mathbf{5}$ ...... (36.9°, 53.1°, 90°) • $1, \sqrt{3}, \mathbf{2},$ ...... (30°, 60°, 90°) • $1, 1, \sqrt{\mathbf{2}}$ ...... (45°, 45°, 90°)

1. Trapezium

A **trapezium** is a quadrilateral which has one pair of parallel sides.



The area of a trapezium $= ah_{\text{parallelogram}} + \frac{1}{2}(b - a)\,h_{\text{triangle}}$

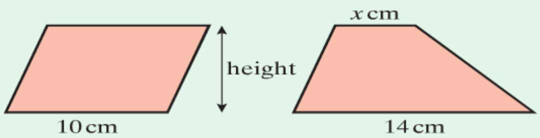
$= ah + \frac{1}{2}bh - \frac{1}{2}ah$

$= \frac{1}{2}ah + \frac{1}{2}bh = \left(\frac{a+b}{2}\right)h$

= half the sum of the lengths of the parallel sides times the height.

Example 1

If a parallelogram has a base of 10 cm, and a trapezium of the same area and height has a base of 14 cm, find x, the length of the other parallel side of the trapezium.



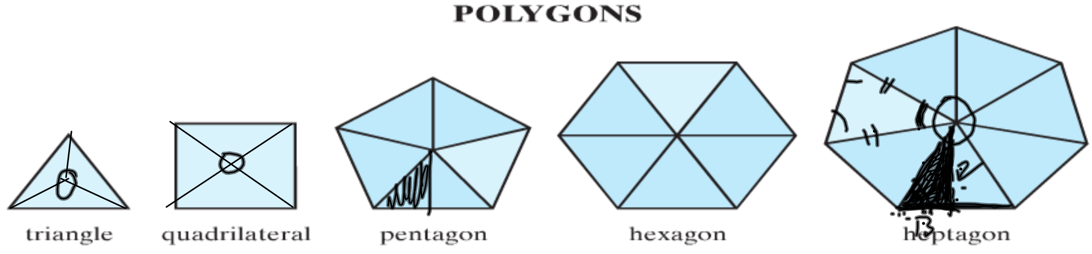
Area Par. = Area Trap.

$Bh = \left(\frac{a+b}{2}\right)h$

$10\cancel{h} = \left(\frac{x+14}{2}\right)\cancel{h}$

$20 = x + 14$

$6 = x$

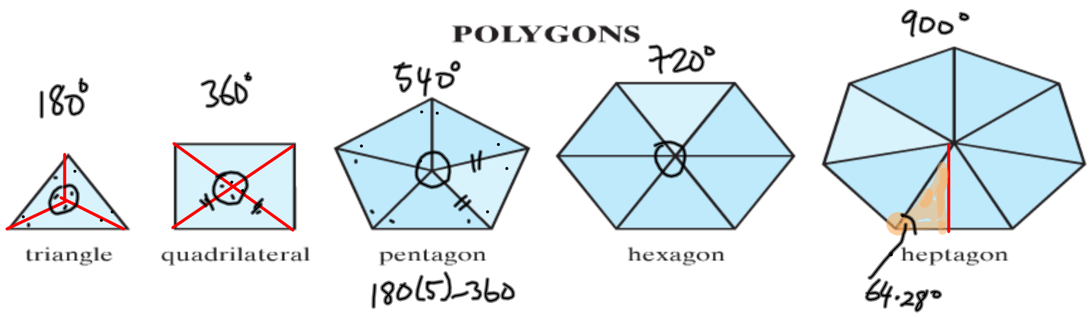


2. Polygons

A **polygon** is a plane (2-dimensional) shape with straight edges.

Regular polygons are symmetrical, with a base triangle repeated in polygons with more than 4 sides. The interior angles of regular polygons are:

Triangle = 60°, Quadrilateral = 90°, Pentagon= 108°, Hexagon = 120°, Heptagon = 128.6°



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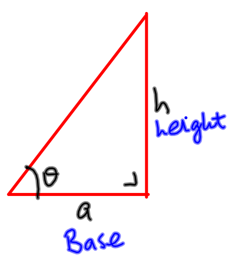
Triangle = 60°, Quadrilateral = 90°, Pentagon= 108°, Hexagon = 120°, Heptagon = 128.6°

We can use the area of a right-angled triangle section to calculate the entire area of a regular polygon.



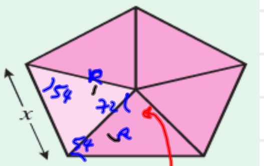
Area of triangle, $\Delta = \frac{Bh}{2}$

$\tan\theta = \frac{h}{a} \Rightarrow h = a\tan\theta$

$\Rightarrow \Delta = \frac{1}{2}a^2\tan\theta$

Example 2

The area of the regular pentagon shown here is 600 cm^2. Calculate the length of one side, x, of the pentagon.



$\frac{360^\circ}{5} = 72^\circ$

Pentagon Area: $A = 600$

1 Triangle Area: $\Delta = \frac{600}{5} = 120 \text{ cm}^2$

Area Triangle

$\Delta = \frac{1}{2} ab \sin C$

$\Rightarrow \frac{R^2 \sin 72}{2} = 120$

$R^2 = \frac{2(120)}{\sin 72} = 252.35$

$R = 15.9$

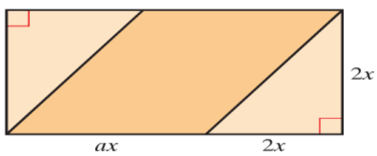
$\Delta = \frac{1}{2} ab \sin C$

$\frac{1}{2}(15.9) x \sin 54 = 120$

$x = \frac{120(2)}{(15.9)\sin 54} = 18.7 \text{ cm}$

Exercise 6.1

1. A parallelogram is drawn inside a rectangle as shown.
Using the measurements given, find
(i) the fraction, in terms of a, of the rectangle's area that is taken up by the parallelogram
(ii) the value of a required so that the area of the parallelogram $= \frac{4}{5}$ the area of the rectangle.



(i)

rectangle
$A = LB$

Total area $= (ax + 2x)(2x) = 2x^2(a+2)$

Parallelogram
$A = Bh$

Parallelogram area $= (ax)(2x) = 2x^2(a)$

Fraction $= \frac{\text{Parallelogram area}}{\text{Total area}} = \frac{\cancel{2x^2}(a)}{\cancel{2x^2}(a+2)} = \frac{a}{a+2}$

(ii)

$\frac{\text{Parallelogram area}}{\text{Total area}} = \frac{a}{a+2} = \frac{4}{5}$

$\Rightarrow 5a = 4a + 8$

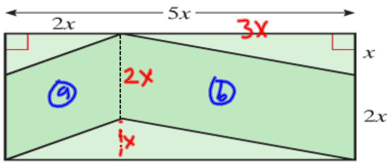
$a = 8$

2. Calculate, in terms of x,

(i) the area of the darker section of the rectangle

(ii) the area of the lighter section of the rectangle

(iii) "the ratio of these areas."



Parallelogram

$A = Bh$

$A_a = 2x(2x) = 4x^2$

$A_b = 2x(3x) = 6x^2$

DARK SECTION $= 10x^2$

Rectangle

$A = LB$

$A = 5x(3x) = 15x^2$

LIGHT SECTION $= 15x^2 - 10x^2 = 5x^2$

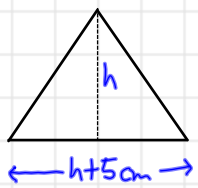
RATIO

DARK : LIGHT

2 : 1

3. If the height of a triangle is 5 cm less than the length of its base, and if the area of the triangle is 52 cm², find the length of the base and also the height of the triangle.

Sketch



Area $= 52 \text{ cm}^2$

$\Delta = \frac{Bh}{2}$

$\Rightarrow \quad 52 = \frac{(h+5)(h)}{2}$

$\Rightarrow \quad 104 = h^2 + 5h$

$h^2 + 5h - 104 = 0$

$(h+13)(h-8) = 0$

$h = -13$ OR 8 cm

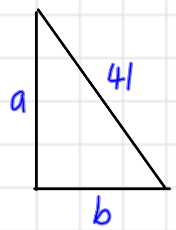
$h = -13$ makes no sense

$\Rightarrow$ height $= 8$ cm

base $= 8+5 = 13$ cm

4. Given that the hypotenuse of a right-angled triangle is 41 cm, and the sum of the sides of the triangle is 49 cm, find the lengths of the other two sides.

Pythagoras



$a^2 + b^2 = 41^2$

$a^2 + b^2 = 1681$ ①

Solve

$a + b = 49$

$\Rightarrow a = 49 - b$ ②

② → ①

$(49 - b)^2 + b^2 = 1681$

$2401 - 98b + b^2 + b^2 = 1681$

$720 - 98b + 2b^2 = 0$

$b^2 - 49b + 360 = 0$

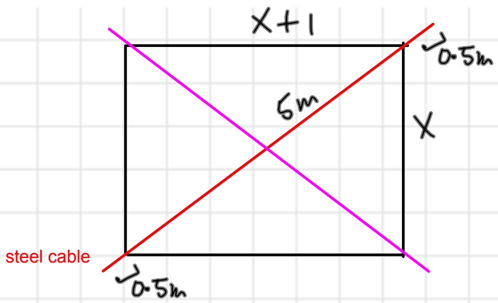
$(b - 9)(b - 40) = 0$

$\Rightarrow b = 9$ OR 40

$\Rightarrow a = 49 - 9 = 40$ OR $a = 49 - 40 = 9$

$\Rightarrow$ Other sides are 9 cm and 40 cm

5. A rectangular wooden frame is built to lay a concrete foundation for a patio. To support the frame while the concrete is being poured, steel cables are fixed diagonally across the rectangle and protrude out of the frame by 50 cm. If the perimeter of the frame is 14 m, and the length of the frame is one metre longer than its width, find the length of steel cable required.



P = 14

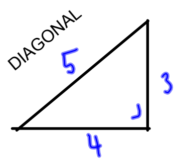
$P = 2(L+B)$

$\Rightarrow 2(x + (x+1)) = 14$

$2x + 1 = 7$

$2x = 6$

$x = 3\text{m}$



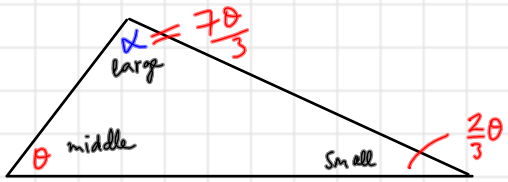
Diagonal = 5m

Each Cable $= 5 + 0.5 + 0.5 = 6\text{m}$

2 cables $= 2(6) = 12\text{m}$

6. In a scalene triangle, the smallest angle is two thirds the size of the middle angle, and the middle angle is three sevenths the size of the largest angle. Find the measure of all three angles.

let middle angle $= \theta$
let large angle $= \alpha$



$$\frac{3}{7}\alpha = \theta \Rightarrow \alpha = \frac{7\theta}{3}$$

180° in triangle

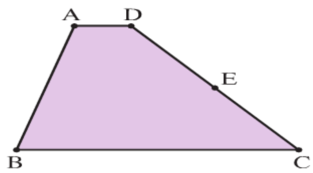
$$\theta + \frac{7\theta}{3} + \frac{2\theta}{3} = 180°$$

×3

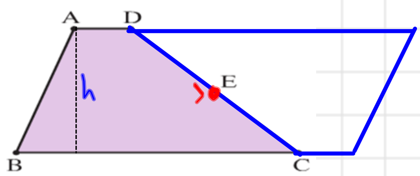
$$\Rightarrow 3\theta + 7\theta + 2\theta = 540°$$
$$12\theta = 540°$$
$$\theta = 540/12 = 45°$$

middle $= 45°$
Small $= (2/3)(45) = 30°$
large $= (7/3)(45) = 105°$

7. E is the midpoint of [DC].
Draw the image of the trapezium ABCD rotated by 180° about the point E.
(i) What shape is made by the image and the original trapezium?
(ii) What is the area of this composite shape?
(iii) Explain how this proves the formula for the area of a trapezium.



Rotation



(i) Parallelogram

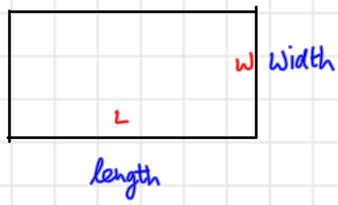
Composite (ii) Area $= (|BC| + |AD|)h$

trapezium (iii) Area $= \frac{1}{2}(\text{composite shape}) = \frac{(|BC| + |AD|)h}{2}$

which is the formula

8. Three times the width of a certain rectangle exceeds twice its length by 3 cm, and four times its length is 12cm more than its perimeter.
Find the dimensions of the rectangle.

let W=width
L=length



$3W = 2L+3$ ①

$P = 2(L+W)$

$4L = P+12$

$4L = 2(L+W)+12$

$2L = L+W+6$

$L = W+6$ ②

Solve ②→①

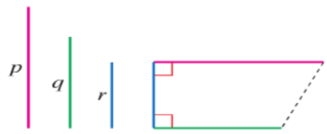
$3W = 2(W+6)+3$

$3W = 2W+12+3$

$W = 15$ cm

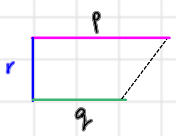
$L = 15+6 \Rightarrow L = 21$ cm

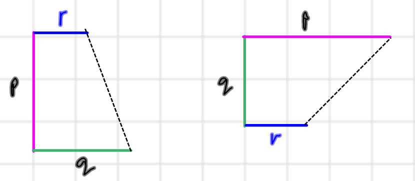
9. Peter has three narrow rods of lengths p, q, and r, where $|p| > |q| > |r|$.
He wants to make a trapezium shape with two right angles as shown. The dotted line completes the trapezium.
Draw the three possible arrangements of rods.
Given the inequality above, find which arrangement has the greatest area.
(Note: if $a > b$, then $ac > bc$, given $c > 0$)



Trapezium area $= \frac{(a+b)h}{2}$

arrangement 1 | arrangement 2 | arrangement 3





Areas

$A_1 = \frac{(p+q)r}{2}$ $\quad A_2 = \frac{(r+q)p}{2}$ $\quad A_3 = \frac{(p+r)q}{2}$

$A_1 = \frac{pr+qr}{2}$ $\quad A_2 = \frac{pr+pq}{2}$ $\quad A_3 = \frac{pq+rq}{2}$

Since $p > r$
$\Rightarrow pq > qr$

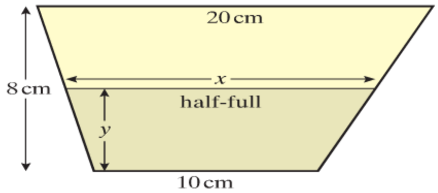
$\Rightarrow \frac{pr+qr}{2} < \frac{pr+pq}{2} \Rightarrow A_1 < A_2$

Since $p > q$
$\Rightarrow pr > rq$

$\Rightarrow \frac{pr+pq}{2} > \frac{pq+rq}{2} \Rightarrow A_2 > A_3$

Conclusion: A_2 is greatest

10. A drawing of the cross-section of a skip is shown. The skip company wants to draw a line along the side of the skip indicating half-full.
Using the dimensions given, find the
(i) length of the line x and
(ii) the height, y, of the line above the base.



Volume relates directly cross sectional area.

Trapezium Area $= \frac{(a+b)}{2}.h$

Full area $= \frac{(20+10)8}{2} = 120 \text{ cm}^2$

$\Rightarrow$ Half area $= 120/2 = 60 \text{ cm}^2$

Consider top and bottom trapezium with equal areas

Area bottom $= \frac{(10+x)y}{2} = 60$

$\Rightarrow (10+x)y = 120 \Rightarrow y = \frac{120}{10+x}$ ①

Area top $= \frac{(20+x)}{2}(8-y) = 60$

$(20+x)(8-y) = 120$

$160 - 20y + 8x - xy = 120$

$-20y + 8x - xy = -40$

$20y - 8x + xy = 40$ ②

10. A drawing of the cross-section of a skip is shown. The skip company wants to draw a line along the side of the skip indicating half-full.
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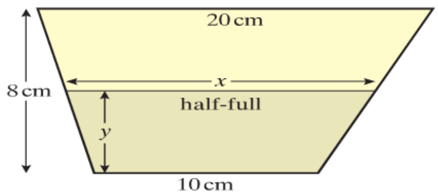


Continued.-

Solve

① → ②

$20\left(\frac{120}{10+x}\right) - 8x + x\left(\frac{120}{10+x}\right) = 40$

$2400 - \cancel{80x} - 8x^2 + \cancel{120x} = 400 + \cancel{40x}$

$+8x^2 = +2000$

$x^2 = 250 \Rightarrow x = 5\sqrt{10} \approx 15.8 \text{ cm}$

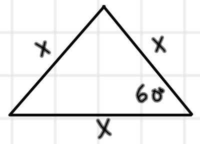
$y = \frac{120}{10+5\sqrt{10}} = 4\sqrt{10} - 8 \approx 4.6 \text{ cm}$

11. (i) The area of an equilateral triangle is 173 cm². Find the length of one side.
(ii) The length of one side of an equilateral triangle is 10.75 cm.
Find the perpendicular height of the triangle and hence find the area of the triangle.
Verify your answer by using the triangle area formula $\frac{1}{2}ab \sin C$.

(i)

let side $= x$

$\Delta = \frac{1}{2}ab\sin C$



$$\Delta = \frac{1}{2}x^2 \sin 60°$$

$$173 = \frac{x^2\sqrt{3}}{4}$$

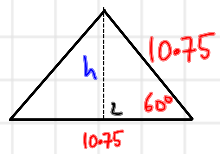
$$\Rightarrow x^2 = \frac{173(4)}{\sqrt{3}} \approx 400$$

$$x = 20 \text{ cm}$$

11. (i) The area of an equilateral triangle is 173 cm². Find the length of one side.
(ii) The length of one side of an equilateral triangle is 10.75 cm.
Find the perpendicular height of the triangle and hence find the area of the triangle.
Verify your answer by using the triangle area formula $\frac{1}{2}ab \sin C$.

(i)

$h = ?$



$$\sin 60° = \frac{h}{10.75}$$

$$h = 10.75 \sin 60° \approx 9.31 \text{ cm}$$

$\Delta = \frac{Bh}{2}$

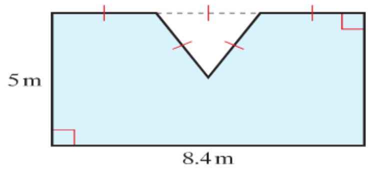
$$\Delta = \frac{(10.75)(9.31)}{2} \approx 50 \text{ cm}^2$$

Verify

$\Delta = \frac{1}{2}ab\sin C$

$$\Delta = \frac{1}{2}(10.75)(10.75)\sin 60° \approx 50 \text{ cm}^2$$

12. Find the area of this figure in square metres, correct to 3 decimal places.



equilateral triangle

$\Delta = \frac{1}{2}ab\sin C$

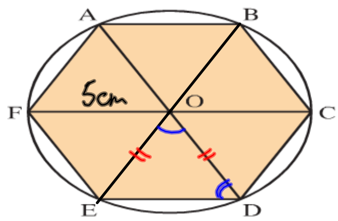
Total area = rectangle − triangle

Triangle Side $= \frac{8.4}{3} = 2.8\text{ m}$

$\Delta = \frac{1}{2}(2.8)(2.8)\sin 60° = 3.3948\text{ m}^2$

Area $= (5)(8.4) - 3.3948 \approx 38.605\text{ m}^2$

13. A regular hexagon is circumscribed by a circle of radius 5 cm. Find
(i) the size of the angle EOD
(ii) the size of the angle ODE
(iii) the area of the hexagon ABCDEFA.



(i) $|\angle EOD| = 360/6 = 60°$

isosceles (ii) $|\angle ODE| = \frac{180-60}{2} = 60°$

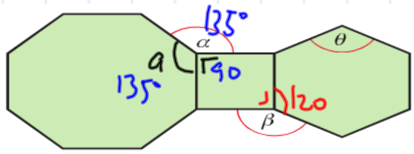
(iii) hexagon has 6 equilateral triangles

$\Delta = \frac{1}{2}ab\sin C$

hexagon area $= 6\left(\frac{1}{2}\right)(5)(5)\sin 60°$

$= 65\text{ cm}^2$

14. A composite design of polygons is shown.
(i) Find the sizes of the angles α, β, θ
(ii) If the square has a side of 4 cm, find the area of this composite shape correct to one place of decimals.



Octogon

Angles in octogon $= 180(8) - 360 = 1080°$

$\Rightarrow a = \frac{1080}{8} = 135°$

$\alpha = 360 - 135 - 90 = 135°$

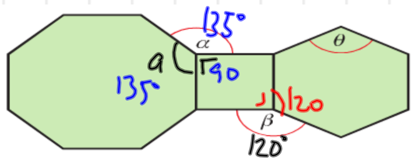
Hexagon

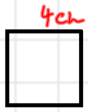
Angles in hexagon $= 180(6) - 360 = 720°$

$\theta = \frac{720°}{6} = 120°$

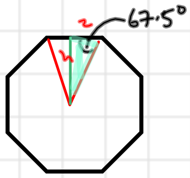
$\beta = 360 - 120 - 90 = 150°$

14. A composite design of polygons is shown.
(i) Find the sizes of the angles α, β, θ
(ii) If the square has a side of 4 cm, find the area of this composite shape correct to one place of decimals.





$A = 4^2 = 16$

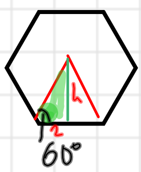


$\tan 67.5 = h/2$

$h = 2\tan 67.5 = 4.82$

$\Delta = \frac{2(4.82)}{2} = 4.82$

Area Octogon $= 16(4.82) = 77.12$



$\tan 60 = h/2$

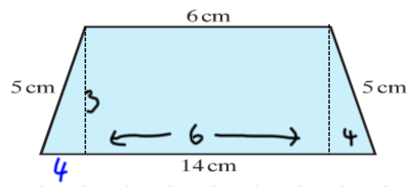
$h = 2\tan 60 = 2\sqrt{3}$

$\Delta = 2(2\sqrt{3})/2 = 2\sqrt{3}$

Area hexagon $= 2\sqrt{3}(12) = 24\sqrt{3}$

Total $= 16 + 77.12 + 24\sqrt{3} = 134.7$ cm^2

15. Using the measurements given, find the area of this trapezium.



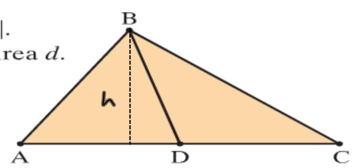
$A = \left(\frac{a+b}{2}\right)h$

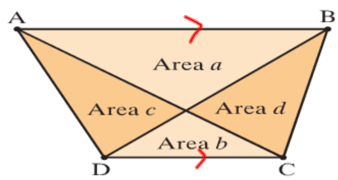
$= \left(\frac{6+14}{2}\right)3 = 30 \text{ cm}^2$

16. (i) Show that the area of $\triangle ABD : \triangle CBD = |AD| : |DC|$.

(ii) ABCD below is a trapezium. Prove that Area c = Area d.

(iii) Hence show that the area of the trapezium ABCD = Area a + Area b $+ 2\sqrt{ab}$.





(i)

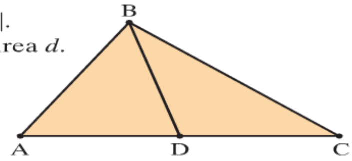
let height $= h$ as shown

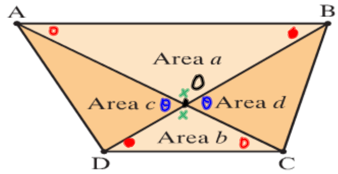
$\Delta = \frac{Bh}{2}$

$\Delta ABD : \Delta CBD$

$\frac{|AD|h}{2} : \frac{|DC|h}{2} = |AD| : |DC|$

16. (i) Show that the area of $\triangle ABD : \triangle CBD = |AD| : |DC|$.
(ii) ABCD below is a trapezium. Prove that Area c = Area d.
(iii) Hence show that the area of the trapezium ABCD = Area a + Area b $+2\sqrt{ab}$.





Let point of diagonal intersection be O.

$|\angle ABD| = |\angle BDC|$ } alternate
$|\angle BAC| = |\angle ACD|$ }
$|\angle AOB| = |\angle DOC|$ (opposite)

$\Rightarrow$ ΔAOB is Similiar to ΔDOC

$\Rightarrow \dfrac{|DO|}{|CO|} = \dfrac{|BO|}{|AO|} \Rightarrow |AO|.|DO| = |BO||CO|$

$|\angle AOD| = |\angle BOC|$ opposite

Area c = Area d

$\Delta = \frac{1}{2} ab \sin C$

$\dfrac{|AO|.|DO|}{2} \sin |\angle AOD| = \dfrac{|BO|.|CO|}{2} \sin |\angle BOC|$ ✓