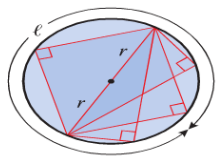
# Area and Volume

# Section 6.2

## Section 6.2 Sectors of circles

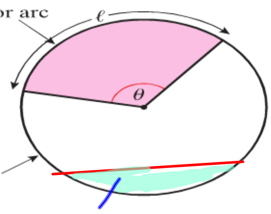
### 1. Revision of circles and sectors of circles

| | | |
|---|---|---|
| Circle/Disc |   | • perimeter (circumference) $= \ell$ ;s ince $\frac{\ell}{2r} = \pi$ $\Rightarrow \ell = 2\pi r$<br>• area $= \pi r^2$<br>• a cyclic quadrilateral is a quadrilateral inscribed in a circle<br>• every triangle inscribed in a semicircle is a right-angled triangle<br>• **$360^o = 2\pi$ radians** |

### 2. Arc of a circle

In the chapter on trigonometry, the length of an arc, the area of a sector, and radian measure were introduced. The length of an arc of a circle is found using the ratios

$$\frac{\ell}{2\pi r} = \frac{\theta\,(\text{degrees})}{360} = \frac{\theta\,(\text{radians})}{2\pi}$$




Segment

$$\therefore \text{ Length of arc } (l) = 2\pi r\frac{\theta\,(\text{degrees})}{360} = 2\pi r\frac{\theta\,(\text{radians})}{2\pi} = r\theta \ \ (\theta \text{ in radians})$$

### 3. Area of a sector

Similarly, the area of a sector of a circle is found using the ratios

$$\frac{A}{\pi r^2} = \frac{\theta\,(\text{degrees})}{360} = \frac{\theta\,(\text{radians})}{2\pi}$$

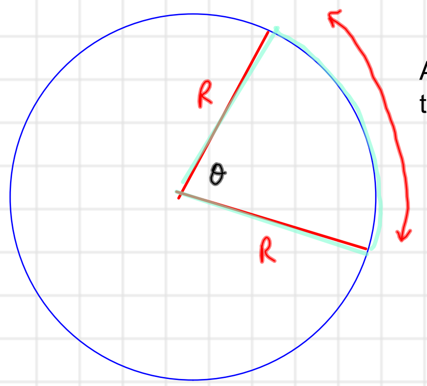
$$\therefore \text{ Area of sector } (A) = \pi r^2\frac{\theta\,(\text{degrees})}{360} = \pi r^2\frac{\theta\,(\text{radians})}{2\pi} = \tfrac{1}{2}r^2\theta \ (\theta \text{ in radians})$$

# Perimeter of Sector

$$\text{Arc length} = \left(\frac{\theta}{360^\circ}\right) 2\pi R$$

fraction

$C = 2\pi R$




An Arc is a fraction of the Circumference

$$\text{Perimeter} = 2r + \text{fraction of } C$$

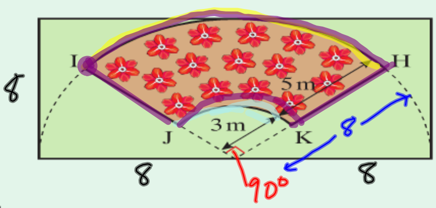
**Example 1**

A flowerbed in the shape of a section of a sector of a circle is placed in the centre of a rectangular lawn, as shown in the diagram. Calculate

(i) the length of edging needed for the flowerbed

(ii) the area of grass in the garden.

Correct each answer to one place of decimals.




$C = 2\pi R$

(i) $\text{Arc}_{IH} = \frac{1}{4}(2\pi 8) = 4\pi$

$\text{Arc}_{JK} = \frac{1}{4}(2\pi 3) = \frac{3}{2}\pi$

Perimeter flowerbed $P = 5 + 5 + 4\pi + \frac{3}{2}\pi = 10 + \frac{11}{2}\pi \approx 27.3\text{ m}$

(ii) Rectangle $A = LB = 8(16) = 128\text{ m}^2$

Small sector $= \frac{1}{4}(\pi(3)^2) = \frac{9}{4}\pi$

large Sector $= \frac{1}{4}(\pi(8)^2) = 16\pi$

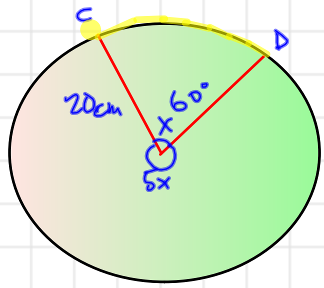
Flower area $= 16\pi - \frac{9}{4}\pi = \frac{55}{4}\pi$

Grass area $= 128 - \frac{55}{4}\pi \approx 84.8\text{ m}^2$

## Example 2

A minor arc CD of a circle, centre O and radius 20 cm, subtends an angle $x$ radians at O. The major arc CD of the circle subtends an angle $5x$ radians at O. Find, in terms of $\pi$, the length of the minor arc.

$C = 2\pi R$



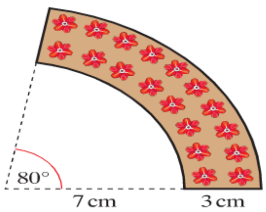

$6x = 360°$

$x = 60°$

$$Arc_{CD} = \frac{1}{6}\,2\pi(20) = \frac{20}{3}\pi$$

## Exercise 6.2

1. A drawing of a curved flower bed is shown. The scale in the drawing is 1 cm : 1 m. Calculate, correct to 1 place of decimals,
   (i) the perimeter of the bed
   (ii) the area of the bed.




(i) Perimeter ?

Flowerbed perimeter has a small arc + large arc + 2 sides of length 3.

Arc length $= \frac{\theta}{360°}(2\pi r)$

$$P = \underbrace{\frac{80}{360}(2)(3.14)(10)}_{\text{large arc}} + \underbrace{\frac{80}{360}(2)(3.14)(7)}_{\text{small arc}} + \underbrace{3+3}_{\text{sides}}$$

$P \approx 29.7$ cm

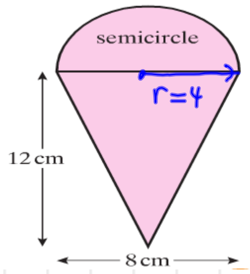
(ii) Area ?

Area Sector $= \left(\frac{\theta}{360°}\right)\pi R^2$

$$A = \underbrace{\frac{80}{360}(3.14)(10)^2}_{\text{large sector}} - \underbrace{\frac{80}{360}(3.14)(7)^2}_{\text{small sector}}$$

$A \approx 35.6$ cm$^2$

**2.** Find:
(i) the total area, correct to the nearest cm²
(ii) the total perimeter enclosed by this composite figure, correct to the nearest cm.



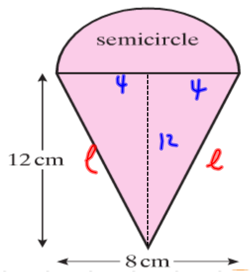

(i) Area

Area Semicircle $= \frac{\pi R^2}{2}$ $= \frac{(3.14)(4)^2}{2} = 25.12\ cm^2$

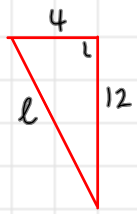
Area triangle $= \frac{Bh}{2}$ $= \frac{12(8)}{2} = 48\ cm$

Total Area $= 48 + 25.12 \approx 73\ cm$ (n.w.n)

**2.** Find:
(i) the total area, correct to the nearest cm²
(ii) the total perimeter enclosed by this composite figure, correct to the nearest cm.




(ii) Perimeter = arc + 2 sides of triangle (ℓ)

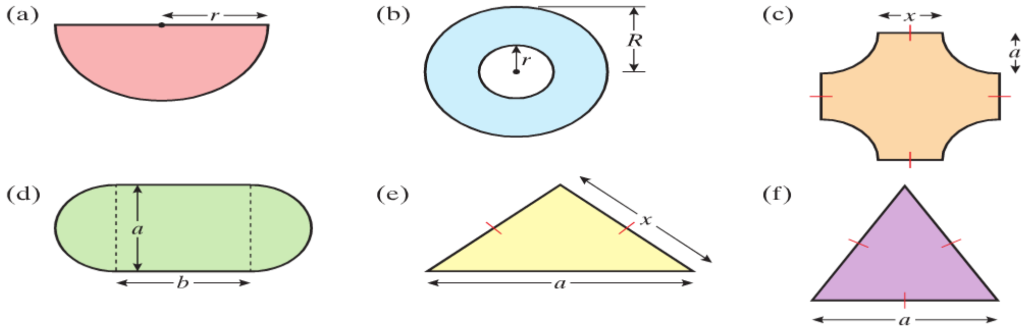



$\ell^2 = 4^2 + 12^2 = 16 + 144$
$\ell = \sqrt{160} = 4\sqrt{10}$

Arc length $= \frac{\theta}{360°}(2\pi r)$ $= \frac{180}{360}(2)(3.14)(4) = 12.56$

$P = 4\sqrt{10} + 12.56 = 25\ cm$ (n.w.n.)

**3.** Write a formula for each of the following shaded areas.




(a) Semicircle Area $= A = \dfrac{\pi r^2}{2}$

(b) Area = large circle − Small circle

$A = \pi R^2 - \pi r^2$

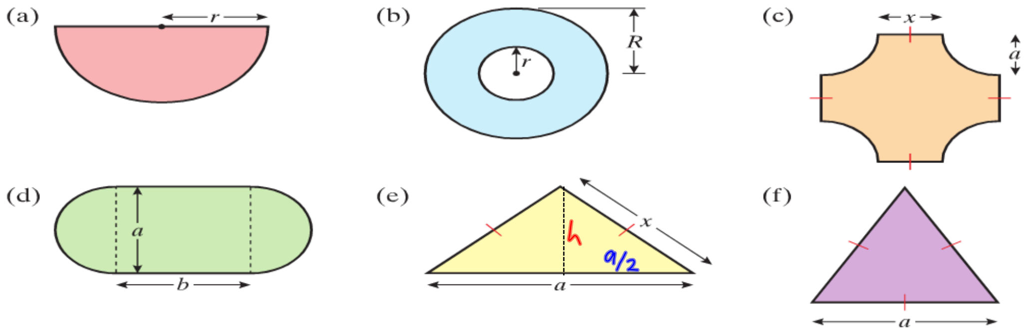
$A = \pi(R^2 - r^2)$

(c) Area = Square − 4 quarters of a circle

Sides = $x+2a$  Radius = $a$

$A = (x+2a)^2 - \pi a^2$

**3.** Write a formula for each of the following shaded areas.




(d) Area = rectangle + 2 semicircles

Sides = $a$ and $b$  radius = $a/2$

$A = ab + \pi\left(\frac{a}{2}\right)^2$

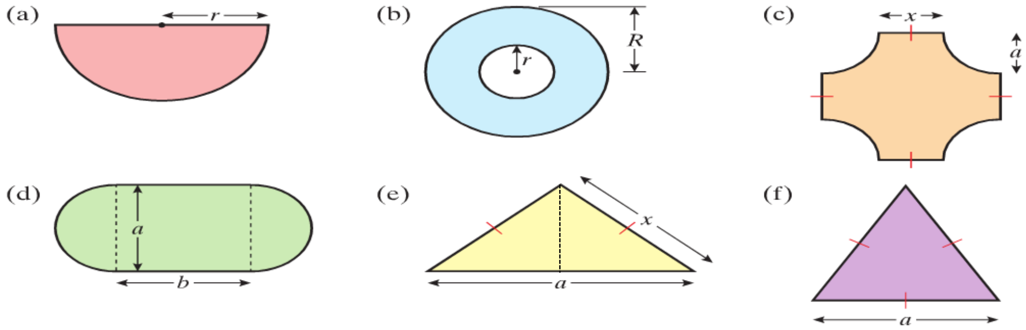
Area $= \dfrac{Bh}{2}$

(e) $h = ?$  $x^2 = \left(\frac{a}{2}\right)^2 + h^2$

$x^2 - \dfrac{a^2}{4} = h^2 \Rightarrow h = \sqrt{x^2 - \dfrac{a^2}{4}}$

$A = \dfrac{a}{2}\sqrt{x^2 - \dfrac{a^2}{4}}$

**3.** Write a formula for each of the following shaded areas.



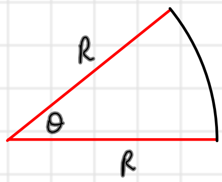

(f) equaliteral triangle

$A = \frac{1}{2}absinC$

$A = \frac{1}{2}(a)(a)\sin 60°$

$\Rightarrow A = \frac{a^2\sqrt{3}}{4}$

**4.** Write a formula for the radius of the sector of a circle in terms of the perimeter P of the sector and the angle $\theta$ radians subtended at the centre.




Arc length $= \frac{\theta}{360}(2\pi R)$

Perimeter of Sector = 2 radii + arc

$P = 2R + \frac{\theta}{360} 2\pi R$

$P = 2R\left(1 + \frac{\theta\pi}{360}\right)$

$R = \frac{P}{2\left(1 + \frac{\theta}{360}\pi\right)}$

**5.** The points R and S lie on the circumference of a circle with centre O and radius 8.5 cm. The point T lies on the major arc RS. Given that $|\angle RTS| = 0.4$ radians, calculate the length of the minor arc RS.

$\pi$ rad. $= 180^\circ$

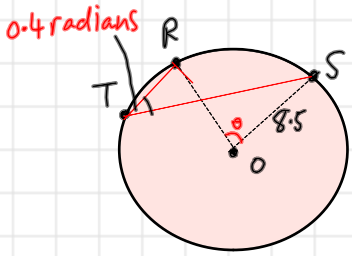
$0.4$ rad $= \frac{0.4}{\pi}(180^\circ)$

$\approx 23^\circ$

in Radians

Arc length $= \frac{\theta}{\cancel{2\pi}}(\cancel{2\pi} R)$

$\Rightarrow$ Arc $= \theta R$




Angle at centre is twice angle on circle standing on the same arc

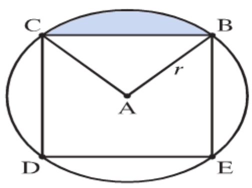
$\Rightarrow \theta = 2|\angle RTS| = 2(0.4) = 0.8$ rad.

minor arc RS $= (0.8)(8.5) = 6.8$ cm

**6.** A square is inscribed inside a circle of radius $r$. Find

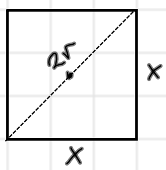
(i) the area of the square BCDE

(ii) the shaded area in terms of $r$.




(i) Area = LB

Pythagoras




$(2r)^2 = x^2 + x^2$

$4r^2 = 2x^2$

$x^2 = 2r^2$

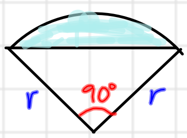
$x = r\sqrt{2}$

Area $= (r\sqrt{2})^2 = 2r^2$

Sector $= \left(\frac{\theta}{360^\circ}\right)\pi R^2$

Shaded Area = Sector − Triangle

* triangle is $\frac{1}{4}$ of square with angle $90^\circ$

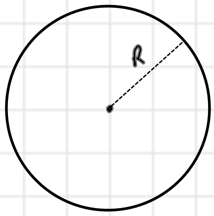



$A = \frac{1}{4}\pi r^2 - \frac{1}{2}(r)(r)$

$A = r^2\left[\frac{\pi}{4} - \frac{1}{2}\right]$

**7.** A farmer has 80 metres of fencing to make a circular chicken coop. Find the radius and the area of the coop to a suitable degree of accuracy. Explain why the radius cannot be measured with complete accuracy.

$C = 2\pi R$



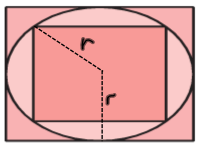

$C = 80\,m$

$\Rightarrow \quad 80 = 2\pi R$

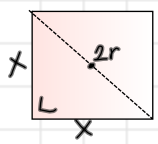
$R = \dfrac{40}{\pi} \approx \dfrac{40}{3.14} = 12.73\ m \quad (2\ d.p.)$

R cannot be measured accurately because $\pi$ is irrational and $R = 40/\pi$ is also irrational.

**8.** (i) A circle is shown with both an inscribed and a circumscribed square. Find the ratio of the area of the inner square to the area of the outer square.



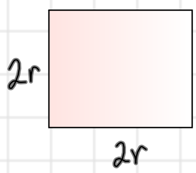

Inner




$(2r)^2 = x^2 + x^2$
$4r^2 = 2x^2$
$2r^2 = x^2$
$x = r\sqrt{2}$

Area $= x^2 = 2r^2$

outer



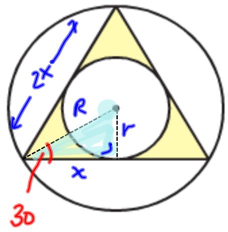

Area $= (2r)^2 = 4r^2$

Ratio

Inner : Outer
$2r^2 : 4r^2$
$1 : 2$

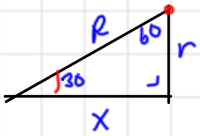
(ii) An equilateral triangle is shown with both an inscribed and a circumscribed circle.

Calculate the ratio of the area of the circumcircle to the area of the incircle.




$A = \pi r^2$

let side of triangle be 2X




How are R and r related?

$\sin 30° = \frac{r}{R} = \frac{1}{2}$

$2r = R$

Ratio

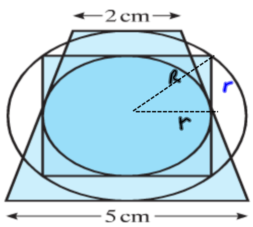
Circumcircle : incircle

$\pi (2r)^2 : \pi r^2$

$4\pi r^2 : \pi r^2$

$4 : 1$

**9.** A circle of circumference 12 cm is inscribed in a square which in turn is inscribed in an outer circle. This outer circle touches the parallel sides of a trapezium as shown.
Find the area of the trapezium, giving your answer in the form $\frac{a\sqrt{b}}{\pi}$.




Trapezium

$A = \frac{(a+b)}{2} h$

$a = 2,\ b = 5,\ h = 2R$

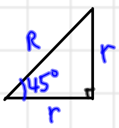
let: Radius of large circle be R
Radius of small circle be r

$A = \frac{(2+5)}{2} 2R = 7R$

$C = 2\pi r$

$2\pi r = 12 \Rightarrow r = \frac{6}{\pi}$

R in terms of r




$R^2 = r^2 + r^2 = 2r^2$

$R = r\sqrt{2}$

Area $\Rightarrow A = 7R = 7\left(\frac{6}{\pi}\right)\sqrt{2} = \frac{42\sqrt{2}}{\pi}$

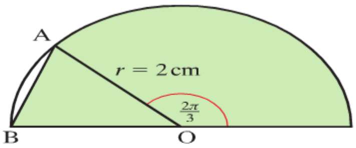
**10.** The shaded portion of the semicircle is to be cut from a large sheet of metal.

(i) Write down in radians the measure of the angle AOB.

(ii) Find the **exact** length of the perimeter of the shaded portion.

(iii) Find the area of the shaded portion and

(iv) hence find the area of the non-shaded segment.

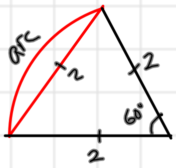



$180° = \pi$ Radians

$\angle AOB = \pi - \frac{2\pi}{3} = \frac{\pi}{3}$

note: $\frac{\pi}{3} = 60°$

Perimeter of non-shaded region?




*notice equilateral triangle ⇒ other side is 2 cm

$Arc = \left(\frac{\theta}{360°}\right) 2\pi R$

Arc length $= \left(\frac{1}{6}\right) 2\pi(2) = \frac{2}{3}\pi$

Perimeter $= \frac{2}{3}\pi + 2$

Area of shaded region?

$Sector = \left(\frac{\theta}{360°}\right)\pi R^2$

$\Delta = \frac{1}{2}ab\sin C$

$Area_s =$ 120° Sector + triangle

$= \left(\frac{2}{6}\right)\pi(2)^2 + \frac{1}{2}(2)(2)\sin 60° = \frac{4}{3}\pi + \sqrt{3}$

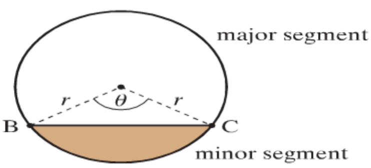
Area of non-shaded region?

$A_{circle} = \pi R^2$

$Area_N =$ Area Semi-circle − Area shaded

$= \frac{1}{2}\pi(2)^2 - \left[\frac{4}{3}\pi + \sqrt{3}\right] = \frac{2\pi}{3} - \sqrt{3}$

**11.** Derive a formula in terms of $r$ and $\theta$ radians for the area of the minor segment under the chord BC. Hence, find the ratio of the area of the major segment to the area of the minor segment subtended by an angle of $\frac{\pi}{2}$ radians.




Area minor segment = Area sector - Area triangle

Sector area (radians) $= \frac{\theta \pi r^2}{2\pi}$

$\Delta = \frac{1}{2}ab\sin C$

$= \frac{\theta}{2\cancel{\pi}}\cancel{\pi} r^2 - \frac{r^2\sin\theta}{2} = \frac{\theta r^2}{2} - \frac{r^2\sin\theta}{2}$

Area minor $= \frac{r^2}{2}(\theta - \sin\theta)$

If $\theta = \frac{\pi}{2}$

Area minor $= \frac{r^2}{2}\left[\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{2}\right)\right] = \frac{r^2}{2}\left(\frac{\pi}{2} - 1\right) = r^2\left(\frac{\pi}{4} - \frac{1}{2}\right)$

Area major segment = Area circle - Area minor segment

$A_{circle} = \pi r^2$

$= \pi r^2 - r^2\left(\frac{\pi}{4} - \frac{1}{2}\right) = r^2\left[\pi - \frac{\pi}{4} + \frac{1}{2}\right] = r^2\left(\frac{3\pi}{4} - \frac{1}{2}\right)$

Ratio

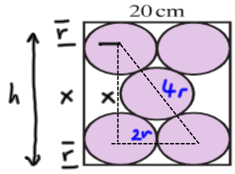
major : minor segment area

$\cancel{r^2}\left(\frac{3\pi}{4} - \frac{1}{2}\right) : \cancel{r^2}\left(\frac{\pi}{4} - \frac{1}{2}\right)$

×4

$3\pi - 2 : \pi - 2$

**12.** Five discs fit exactly into a rectangular frame of width 20 cm. Find the area of the remaining space in the frame.

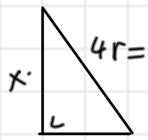



$A_{circle} = \pi R^2$

$R = \frac{20}{4} = 5$ cm

$A_{circle} = \pi(5)^2 = 25\pi$

$\Rightarrow$ 5 circles area $= 5(25\pi) = 125\pi \approx 392.7$ cm$^2$




$x^2 = 20^2 - 10^2$

$= 400 - 100 = 300$

$x = \sqrt{300} = 10\sqrt{3}$

$h = 2r + x = 2(5) + 10\sqrt{3} = 10 + 10\sqrt{3}$

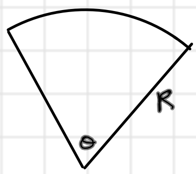
$A = LB$

Frame area = $20(10 + 10\sqrt{3}) \approx 546.4$ cm$^2$

Empty space $= 546.4 - 392.7 = 153.7$ cm$^2$

**13.** If the area of a sector of a circle is 48 cm$^2$, and its perimeter is 28 cm, find the length of the radius.

does this mean Perimeter of Sector or circle?




$P = 28$
$A = 48$
$R = ?$

$A = \frac{\theta}{360}\pi R^2$

$P = 2R + \frac{\theta}{360}2\pi R$

Perimeter: $28 = 2R + \frac{\theta}{360°}2\pi R$

$\Rightarrow \frac{28 - 2R}{2\pi R} = \frac{\theta}{360°} = \frac{14 - R}{\pi R}$

Area: $48 = \frac{\theta}{360°}\pi R^2 \Rightarrow \frac{48}{\pi R^2} = \frac{\theta}{360°}$

$\Rightarrow \frac{14 - R}{\cancel{\pi R}} = \frac{48}{\cancel{\pi R^2}} \Rightarrow R(14 - R) = 48$

$14R - R^2 - 48 = 0$

$R^2 - 14R + 48 = 0$

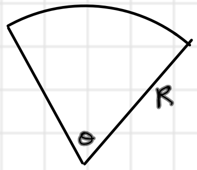
$(R - 6)(R - 8) = 0$

$\Rightarrow R = 6$ or $8$ cm depending on θ

**13.** If the area of a sector of a circle is 48 cm², and its perimeter is 28 cm, find the length of the radius.

does this mean Perimeter of Sector or circle?

Continued...




$P = 28$
$A = 48$

$\Rightarrow R = 6$ or $8$ cm depending on $\theta$

Since $\dfrac{48}{\pi R^2} = \dfrac{\theta}{360^\circ} \Rightarrow \theta = \dfrac{48(360)}{\pi R^2}$

if $R=6$ $\Rightarrow$ $\theta = \dfrac{48(360)}{\pi(6)^2} = \dfrac{480}{\pi}$

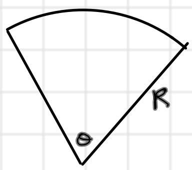
check Area $= \dfrac{\left(\frac{480}{\cancel{\pi}}\right)}{360}(\cancel{\pi})(6)^2 = 48$ ✓

Perimeter $= 2(6) + \dfrac{\left(\frac{480}{\cancel{\pi}}\right)}{360} 2\cancel{\pi}(6) = 28$ ✓

**13.** If the area of a sector of a circle is 48 cm², and its perimeter is 28 cm, find the length of the radius.

does this mean Perimeter of Sector or circle?

Continued...




$P = 28$
$A = 48$

$\Rightarrow R = 6$ or $8$ cm depending on $\theta$

Since $\dfrac{48}{\pi R^2} = \dfrac{\theta}{360^\circ} \Rightarrow \theta = \dfrac{48(360)}{\pi R^2}$

if $R=$ $\Rightarrow$ $\theta = \dfrac{48(360)}{\pi(8)^2} = \dfrac{270}{\pi}$

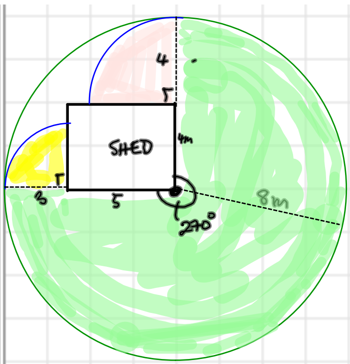
check Area $= \dfrac{\left(\frac{270}{\cancel{\pi}}\right)}{360}(\cancel{\pi})(8)^2 = 48$ ✓

Perimeter $= 2(8) + \dfrac{\left(\frac{270}{\cancel{\pi}}\right)}{360} 2\cancel{\pi}(8) = 28$ ✓

$\Rightarrow r = 6$ cm or $8$ cm depending on $\theta$.

**14.** A farmer has a shed measuring 4 m by 5 m in the centre of a large field of grass. He ties a goat to one corner of this shed, and using a rope measuring 8 m, allows him to graze on the grass.

(i) Draw a diagram showing the grazing area.
(ii) Indicate on the diagram the different sectors of circles represented by this area.
(iii) Calculate this total grazing area correct to the nearest m².




Grazing area includes 3 sectors shown with radii and angles

- $r = 3$, $\theta = 90°$
- $r = 4$, $\theta = 90°$
- $r = 8$, $\theta = 270°$

Sector Area $= \frac{\theta}{360} \pi R^2$

$$\text{Total area} = \frac{1}{4}\pi(3)^2 + \frac{1}{4}\pi(4)^2 + \frac{3}{4}\pi(8)^2$$

$$= \frac{\pi}{4}\left(9 + 16 + 3(64)\right)$$

$$= \frac{217}{4}\pi \approx 170 \text{ m}^2$$