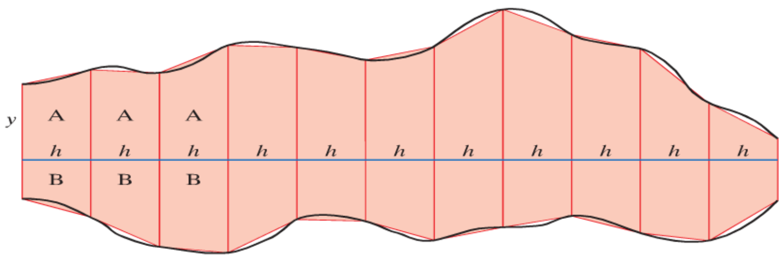
Area and Volume

Section 6.4

Section 6.4 Trapezoidal rule for calculating area

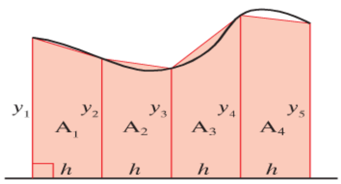
To calculate the areas of shapes with irregular boundaries, e.g. fields, lakes, etc., surveyors have usually divided the area into a series of parallel strips, each in the shape of a trapezium; a quadrilateral with two of the four sides parallel to each other.



A straight line is drawn across the centre of the area, dividing it into a series of two different areas, A and B.

The area of each section, above and below the line, can be calculated separately using the formula for the area of a trapezium and then added together.

Along the line and at equal intervals of h, perpendicular lines are drawn up to the boundary. These ordinates (offsets) – y_1, y_2, y_3, etc – are the parallel sides of the trapezium.



Using the area formula for a trapezium, $\frac{a+b}{2} \times h$, we get

$A_1 = \frac{y_1 + y_2}{2} \times h$. Similarly, $A_2 = \frac{y_2 + y_3}{2} \times h$, and so on.

Therefore, the total area $A = A_1 + A_2 + A_3 + A_4$.

$$= \left(\frac{y_1 + y_2}{2} \times h\right) + \left(\frac{y_2 + y_3}{2} \times h\right) + \left(\frac{y_3 + y_4}{2} \times h\right) + \left(\frac{y_4 + y_5}{2} \times h\right)$$

$$= \frac{h}{2}(y_1 + y_2 + y_2 + y_3 + y_3 + y_4 + y_4 + y_5)$$

$$= \frac{h}{2}[y_1 + 2(y_2 + y_3 + y_4) + y_5]$$

In words, Area $\approx \dfrac{\text{interval width}}{2}$ [first height + last height + 2(remaining heights)]

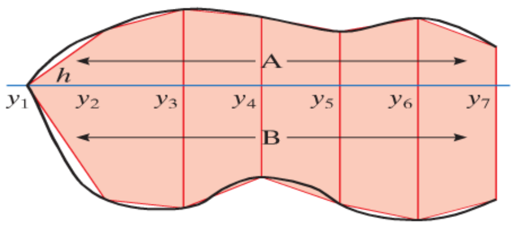
When n strips are made, the Trapezoidal formula becomes

$$\text{Area} \approx \frac{h}{2}\left[y_1 + y_n + 2(\,y_2 + y_3 + y_4 + \dots .\, y_{n-1})\right]$$

Note 1: Because the top of each trapezium does not match the boundary at all points, the area obtained by this formula is only approximate.
Its accuracy depends on the gap width h ; the smaller the gap width, the greater the accuracy.

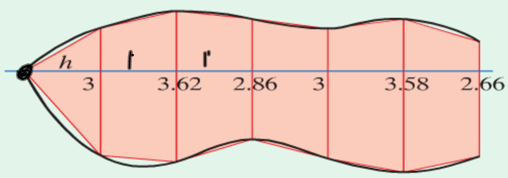
Note 2: If offsets are measured from the same points above and below the line, then the area (A + B) can be obtained using

$$\text{Area} \approx \frac{h}{2}\left[y_1 + y_7 + 2(\,y_2 + y_3 + y_4 + y_5 + y_6)\right]$$



Example 1

Using the measurements provided, find the area of this shape given $h = 1$ unit.



$h = 1$

$y_1 = 0$ $y_2 = 3$ $y_3 = 3.62$

$y_4 = 2.86$ $y_5 = 3$ $y_6 = 3.58$

$y_5 = 2.66$

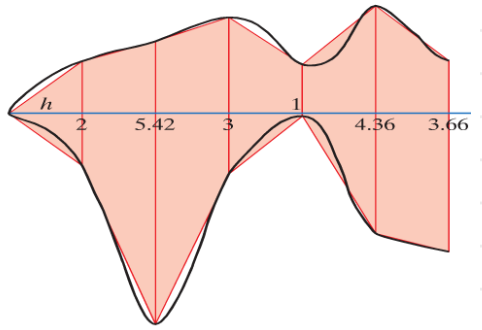
Area $= \frac{1}{2}\left[0 + 2.66 + 2(3 + 3.62 + 2.86 + 3 + 3.58)\right]$

$= 17.3$

2. If $h = 1$ cm and the lengths of the offsets are as shown, find the area of this map.

(i) If the area of the map is 17.23 cm^2, find the percentage error in using the trapezoidal rule and $h = 1$ cm.

(ii) By taking new measurements with $h = \frac{1}{2}$ cm, find a second estimate of the area.



Remember...

Trapoziodal Rule

$\text{Area} \approx \frac{h}{2}\left[\text{First} + \text{Last} + 2(\text{others})\right]$

$h = 1$ cm

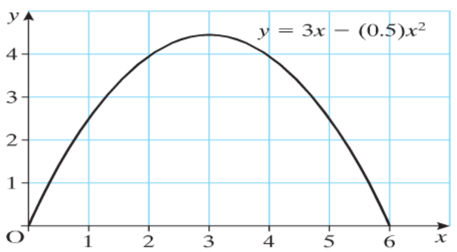
$$\text{Area} \approx \frac{1}{2}\left[0 + 3.66 + 2(2 + 5.42 + 3 + 1 + 4.36)\right]$$

$$\approx 17.61\text{ cm}^2$$

$$\%\text{ ERROR} = \frac{\text{Error} \times 100}{\text{"right result"}}$$

$$= \frac{(17.23 - 17.61) \times 100}{17.23} = 2.2\%$$

3. Using the trapezoidal rule, and an interval value of (i) $h = 1$ cm and (ii) $h = 0.5$ cm, estimate the area under the curve $y = 3x - (0.5)x^2$.



use calculator

X	f(X)
0	0
1	2.5
2	4
3	4.5
4	4
5	2.5
6	0

$h = 1$ cm

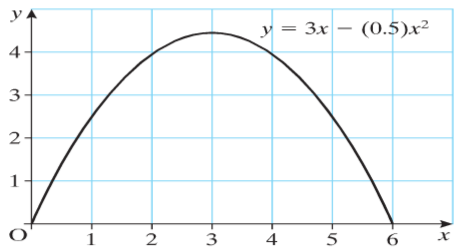
Trapoziodal Rule

$\text{Area} \approx \frac{h}{2}\left[\text{First} + \text{Last} + 2(\text{others})\right]$

$$\text{Area} \approx \frac{1}{2}\left[0 + 0 + 2(2.5 + 4 + 4.5 + 4 + 2.5)\right]$$

$$\approx 17.5\text{ cm}^2$$

3. Using the trapezoidal rule, and an interval value of (i) $h = 1\text{ cm}$ and (ii) $h = 0.5\text{ cm}$, estimate the area under the curve $y = 3x - (0.5)x^2$.



use calculator

X	f(X)
0	0
0.5	1.375
1	2.5
1.5	3.375
2	4
2.5	4.375
3	4.5
3.5	4.375
4	4
4.5	3.375
5	2.5
5.5	1.375
6	0

$h = 0.5\text{ cm}$

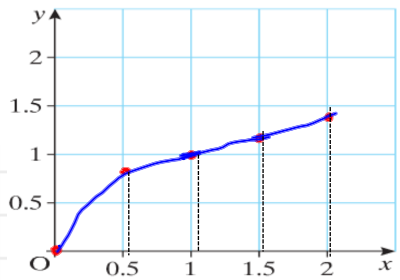
Trapoziodal Rule

$$\text{Area} \approx \frac{h}{2}\left[\text{First} + \text{Last} + 2(\text{others})\right]$$

$$\approx \frac{0.5}{2}\left[0+0+2\left(2(\underbrace{1.375+2.5+3.375+4+4.375}_{\text{repeated numbers}})+4.5\right)\right]$$

$$\approx 17.875 \text{ cm}^2$$

4. Copy these axes and use them to plot the function $y = \sqrt{x}$ for $0 \leqslant x \leqslant 2$. Using four trapezoids, approximate the area under the curve for $0 \leqslant x \leqslant 2$.



X	$y=\sqrt{x}$
0	0
0.5	0.7
1	1
1.5	1.2
2	1.4

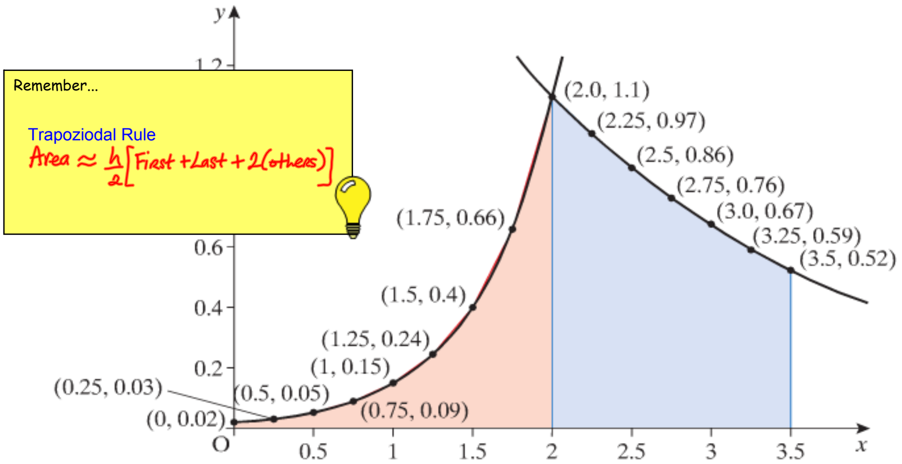
Trapoziodal Rule

$$\text{Area} \approx \frac{h}{2}\left[\text{First} + \text{Last} + 2(\text{others})\right]$$

$$\text{Area} \approx \frac{0.5}{2}\left[0+1.4+2(0.7+1+1.2)\right]$$

$$\approx 1.8$$

5.



(i) Using an interval width of 0.25, find the ratio of the coloured areas under the curve.
(ii) Estimate, using trial and error, the maximum value of x so that both areas are equal.

(i)

Red Area $= \frac{0.25}{2}\left[0.02 + 1.1 + 2(0.03 + 0.05 + 0.09 + 0.15 + 0.24 + 0.4 + 0.66)\right]$

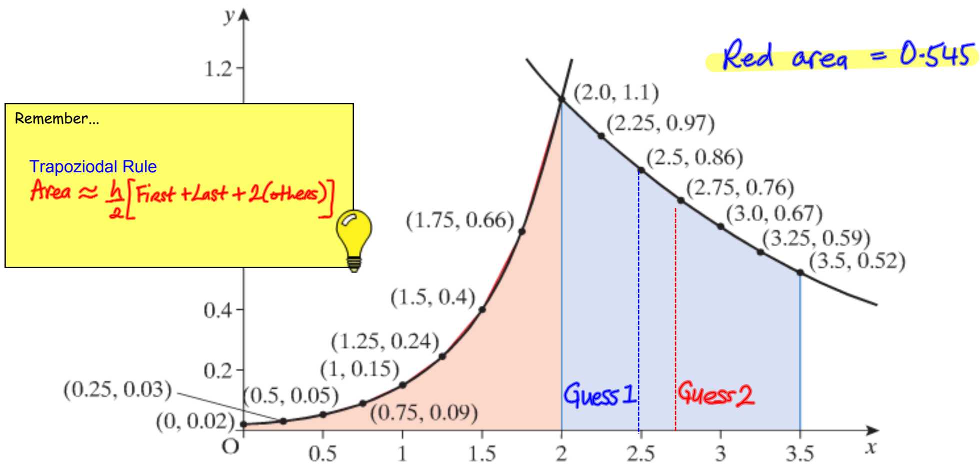
$= 0.545$

Blue Area $= \frac{0.25}{2}\left[1.1 + 0.52 + 2(0.97 + 0.86 + 0.76 + 0.67 + 0.59)\right]$

$= 1.165$

Ratio Red : Blue $= 0.545 : 1.165 = 109 : 233$

5.



(i) Using an interval width of 0.25, find the ratio of the coloured areas under the curve.
(ii) Estimate, using trial and error, the maximum value of x so that both areas are equal.

Guess 1 If max. x is 2.5 then blue area

$= \frac{0.25}{2}\left[1.1 + 0.86 + 2(0.97)\right] = 0.4875$

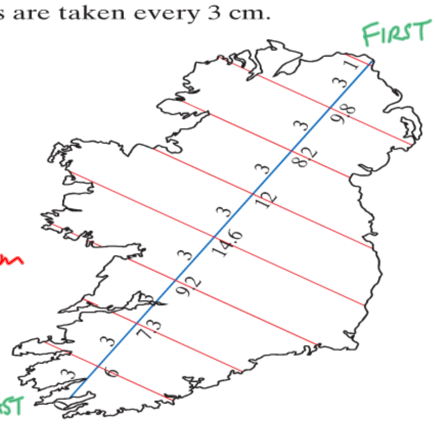
Guess 2 If max x is 2.75 then blue area

$= \frac{0.25}{2}\left[1.1 + 0.76 + 2(0.97 + 0.86)\right] = 0.69$

⇒ Estimate: $x = 2.6$

6. An outline of the map of Ireland is given. If the scale used is 1 cm = 20 km, use the trapezoidal rule to estimate the area of the island of Ireland.
Offsets are taken every 3 cm.

$h = 3\text{cm}$



Remember...

Trapoziodal Rule

$\text{Area} \approx \frac{h}{2}\left[\text{First} + \text{Last} + 2(\text{others})\right]$

$1\text{cm} = 20\text{ km}$

$\Rightarrow 1\text{cm}^2 = 400\text{ km}^2$

Area (cm scale) $\approx \frac{3}{2}\left[1 + 0 + 2(6 + 7.3 + 9.2 + 14.6 + 12 + 8.2 + 9.8)\right]$

$\approx 202.8\text{ cm}^2$

Area (full scale) $\approx 202.8(400) = 81{,}120\text{ km}^2$