6th Year

Maths Test

Solutions

Christmas 1012

Mr. Roche

PART 1 Differentiation

(a) Differentiate $\sqrt{x}\,(x+2)$ with respect to x.

Could use product Rule But easier to multiply out

$$y = (x)^{\frac{1}{2}}(x+2)$$

$$= x^{3/2} + 2x^{1/2}$$

$$\frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}} + x^{-\frac{1}{2}}$$

(b) Parametric equations of a curve are: $x = \frac{2t-1}{t+2}$, $y = \frac{t}{t+2}$, $t \in \mathbb{R} \setminus \{-2\}$

(i) Find $\frac{dy}{dx}$.

(ii) What does the answer in part (i) tell you about the shape of the graph?

Parametric Differentiation

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

Quotient Rule

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

(i)

$\frac{dy}{dt} = ?$ $\quad u = t \quad v = t+2$

$\frac{du}{dt} = 1 \quad \frac{dv}{dt} = 1$

$$\frac{dy}{dt} = \frac{(t+2)(1) - (t)(1)}{(t+2)^2} = \frac{t+2-t}{(t+2)^2} = \frac{2}{(t+2)^2}$$

$\frac{dx}{dt} = ?$ $\quad u = 2t-1 \quad v = t+2$

$\frac{du}{dt} = 2 \quad \frac{dv}{dt} = 1$

$$\frac{dx}{dt} = \frac{(t+2)(2) - (2t-1)(1)}{(t+2)^2}$$

$$= \frac{2t+4-2t+1}{(t+2)^2} = \frac{5}{(t+2)^2}$$

$$\Rightarrow \frac{dt}{dx} = \frac{(t+2)^2}{5}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{2}{\cancel{(t+2)^2}} \cdot \frac{\cancel{(t+2)^2}}{5} = \frac{2}{5}$$

(ii) Constant slope means that it is a straight line graph.

(c) $f(x) = \log_e 3x - 3x$, where $x > 0$ $\log_e = \ln$

(i) Show that $\left(\frac{1}{3}, -1\right)$ is a local maximum point of $f(x)$.

$f(x) \to f'(x)$

$\ln x \to \frac{1}{x}$

at maximum

$f'(x) = 0$

and $f''(x) < 0$

$f(x) = \ln 3x - 3x$

$f'(x) = \left(\frac{1}{3x}\right)(3) - 3 = \frac{1}{x} - 3$

$f'(x) = \frac{1}{x} - 3 = 0$

$\frac{1}{x} = \frac{3}{1} \Rightarrow x = \frac{1}{3}$

Sub into function to find y value

$f\left(\frac{1}{3}\right) = \ln\left(3\left(\frac{1}{3}\right)\right) - 3\left(\frac{1}{3}\right) = \ln 1 - 1 = -1$

$\Rightarrow$ point $\left(\frac{1}{3}, -1\right)$ is a max. OR min.

get $f''(x)$

$f'(x) = x^{-1} - 3$

$f''(x) = -x^{-2}$

$f''\left(\frac{1}{3}\right) = -\left(\frac{1}{3}\right)^{-2} = -9 < 0 \Rightarrow$ maximum.

(ii) Differentiate $\frac{1}{x}$ with respect to x from first principles

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f(x) = \frac{1}{x}$$

$$f(x+h) = \frac{1}{(x+h)}$$

$$f(x+h) - f(x) = \frac{1}{x+h} - \frac{1}{x} = \frac{x - (x+h)}{x(x+h)} = \frac{-h}{x^2+hx}$$

$$\frac{f(x+h) - f(x)}{h} = \frac{-\cancel{h}\,1}{(x^2+hx)\cancel{h}} = \frac{-1}{x^2+hx}$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \frac{-1}{x^2+(0)x} = \frac{-1}{x^2}$$

(a) The equation $x^3 + x^2 - 4 = 0$ has only one Real Root

Taking $x_1 = \frac{3}{2}$ as the first approximation to the Root

use the Newton-Raphson method to find x_2 the second approximation.

Newton-Raphson

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$f(x) = x^3 + x^2 - 4$$

$$f'(x) = 3x^2 + 2x$$

$$f\left(\tfrac{3}{2}\right) = \left(\tfrac{3}{2}\right)^3 + \left(\tfrac{3}{2}\right)^2 - 4 = \frac{13}{8}$$

$$f'\left(\tfrac{3}{2}\right) = 3\left(\tfrac{3}{2}\right)^2 + 2\left(\tfrac{3}{2}\right) = \frac{39}{4}$$

$$x^2 = \frac{3}{2} - \frac{(13/8)}{(39/4)} = \frac{4}{3}$$

(b) The equation of the curve is $y = 3x^4 - 2x^3 - 9x^2 + 8$

(i) show the curve has a local maximum at the point $(0,8)$

(ii) find the coordinates of the two local minimum points or the curve.

(iii) Draw a Sketch of the curve.

(i) at maximum

$\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$

Solve

$$\frac{dy}{dx} = 12x^3 - 6x^2 - 18x$$

at max/min $\Rightarrow$ $12x^3 - 6x^2 - 18x = 0$

$6x(2x^2 - x - 3) = 0$

$6x(2x - 3)(x + 1) = 0$

$\Rightarrow 6x = 0$, $x = 0$ | $2x - 3 = 0$, $x = 3/2$ | $x = -1$

$x = 0$, $y = ?$ $\quad y = 3(0)^4 - 2(0)^3 - 9(0)^2 + 8 = 8$ pt. $(0,8)$

Show $f'(0,8) < 0$ $\quad \left.\frac{d^2y}{dx^2}\right|_{(0,8)} = 36x^2 - 12x - 18 = -18 < 0$

$\Rightarrow (0,8)$ is a maximum

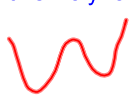
(ii)

other turning pts. $x = 3/2$, $y = ?$ $\quad y = 3\left(\frac{3}{2}\right)^4 - 2\left(\frac{3}{2}\right)^3 - 9\left(\frac{3}{2}\right)^2 + 8 = -\frac{61}{16}$ pt. $\left(3/2, -\frac{61}{16}\right)$

$x = -1$, $y = ?$ $\quad y = 3(-1)^4 - 2(-1)^3 - 9(-1)^2 + 8 = 4$ pt. $(-1, 4)$

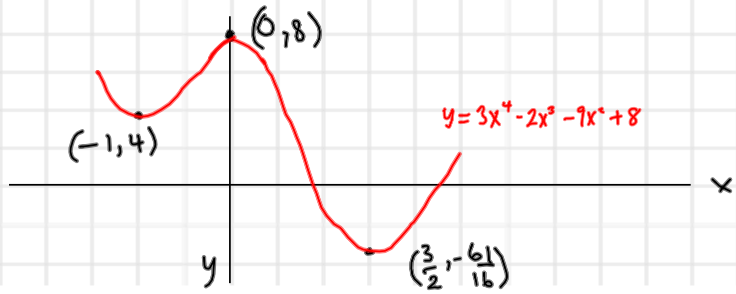
(iii) Sketch

Positive Polynomial with 3 turning pts.



shape

$-\frac{61}{16} \simeq -3.8$



(c) A curve is defined by the equation $\underbrace{x^2y^3}_{\text{Product}} + 4x + 2y = 12$

(i) Find $\frac{dy}{dx}$ in terms of x and y.

(ii) Show that the tangent to the curve at the point (0, 6) is also the tangent to it at the point (3, 0)

(i)

Implicit Differentiation

need to differentiate

each part with respect to x

$u = x^2 \qquad v = y^3$

$\frac{du}{dx} = 2x \qquad \frac{dv}{dx} = 3y^2\frac{dy}{dx}$

Product Rule

$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

EQUATION: $x^2y^3 + 4x + 2y = 12$

↓

DERIVATIVE: $x^2\left(3y^2\frac{dy}{dx}\right) + y^3(2x) + 4 + 2\frac{dy}{dx} = 0$

use algebra to isolate $\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx}\left[3x^2y^2 + 2\right] = -2xy^3 - 4$

$\frac{dy}{dx} = \frac{-2xy^3 - 4}{3x^2y^2 + 2}$

(ii)

Slope at (0, 6)

$\frac{dy}{dx}_{(0,6)} = \frac{-2(0)(6)^3 - 4}{3(0)^2(6)^2 + 2} = \frac{-4}{2} = -2$

Tangent equation: $y - y_1 = m(x - x_1)$

$\Rightarrow y - 6 = -2(x - 0) \Rightarrow 2x + y - 6 = 0$

Slope at (3, 0)

$\frac{dy}{dx}_{(3,0)} = \frac{-2(3)(0)^3 - 4}{3(3)^2(0)^2 + 2} = -2$

Tangent equation: $y - y_1 = m(x - x_1)$

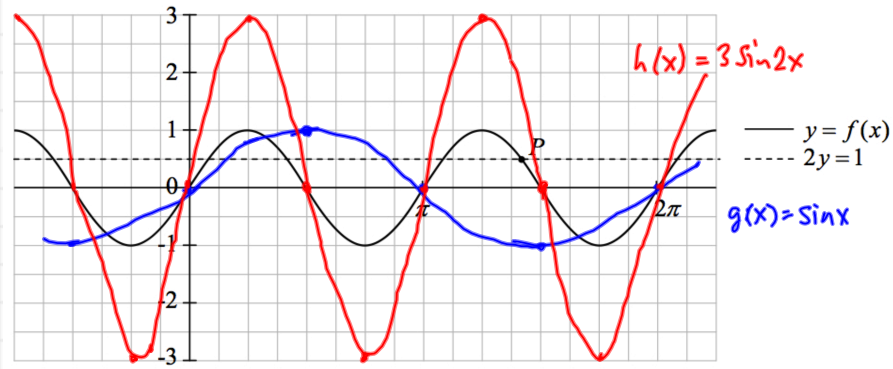
$\Rightarrow y - 0 = -2(x - 3) \Rightarrow 2x + y - 6 = 0$

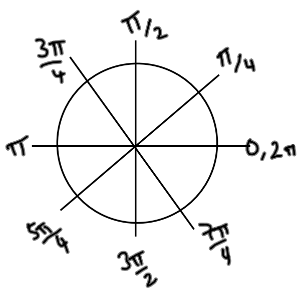
Part 2: Trigonometry

The diagram below shows the graph of the function $f : x \mapsto \sin 2x$. The line $2y = 1$ is also shown.

(a) On the same diagram above, sketch the graphs of $g : x \mapsto \sin x$ and $h : x \mapsto 3\sin 2x$.
Indicate clearly which is g and which is h.

(b) Find the co-ordinates of the point P in the diagram.





Table

x	0	π/4	π/2	3π/4	π	5π/4	3π/2	7π/4	2π
Sin x	0	–	1	–	0	–	−1	–	0
3 sin 2x	0	3	0	−3	0	3	0	−3	0

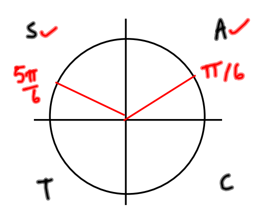
(b)

OBSERVE at P it is the 4th time $g(x) = \frac{1}{2}$

$g(x) = \sin 2x = \frac{1}{2}$

$2x = \sin^{-1}(\frac{1}{2}) = \pi/6 \Rightarrow x = \pi/12$

Sinθ is Positive



2nd time $g(x) = \frac{1}{2} \Rightarrow 2x = 5\pi/6 \Rightarrow x = 5\pi/12$

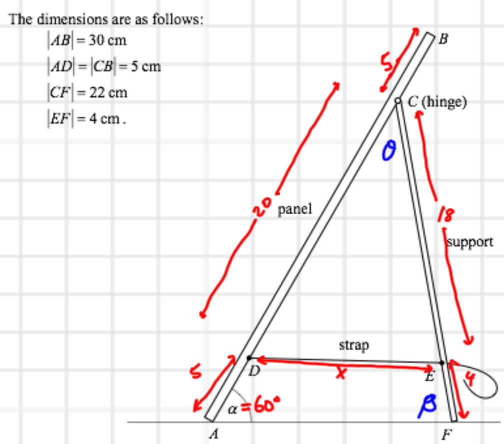
OBSERVE 4th time $g(x) = \frac{1}{2} \Rightarrow x = \frac{5\pi}{12} + \pi = \frac{17\pi}{12}$

Check $\sin\left[2\left(\frac{17\pi}{12}\right)\right] = \frac{1}{2}$ ✓

$\Rightarrow P = \left(\frac{17\pi}{12}, \frac{1}{2}\right)$

(ii) Find the length of the strap [DE] such that the angle α between the panel and the ground is 60°

note [DE] is not parallel to [AF]

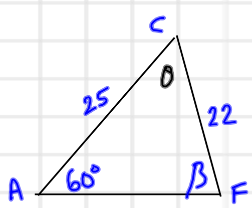


θ is common to 2 triangles

If I could work out angle θ we could solve for x

Use Sine Rule

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$



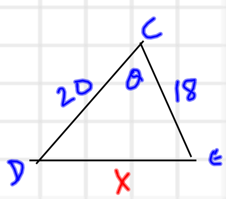
$$\frac{\sin\beta}{25} = \frac{\sin 60°}{22}$$

$$\beta = \sin^{-1}\left(\frac{25\sin 60°}{22}\right) \approx 79.8°$$

$$\theta = 180 - 60 - 79.8 \approx 40.2°$$

Use Cosine Rule

$$a^2 = b^2 + c^2 - 2bc\cos A$$



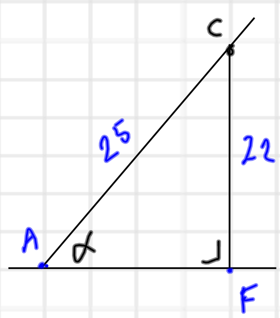
$$x^2 = (20)^2 + (18)^2 - 2(20)(18)\cos(40.2)$$

$$x^2 = 174.1$$

$$x \approx 13.2 \text{ cm}$$

(ii) Find the maximum value of α

α is maximum when panel is most upright



$$\sin \alpha = \frac{22}{25}$$

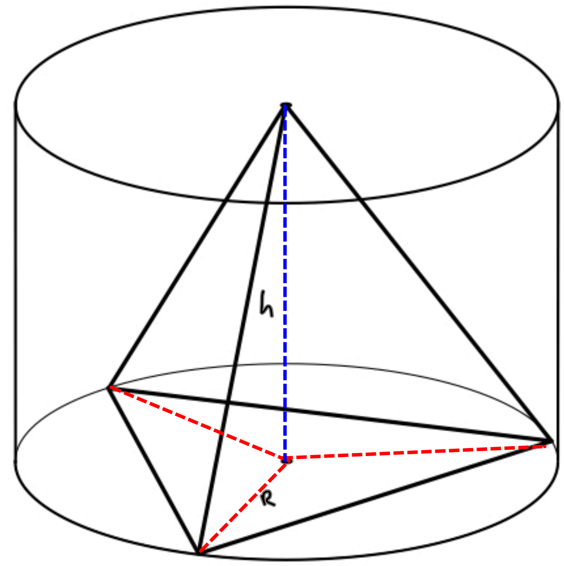
$$\alpha = \sin^{-1}\left(\frac{22}{25}\right)$$

$$\alpha \simeq 61.6^\circ$$

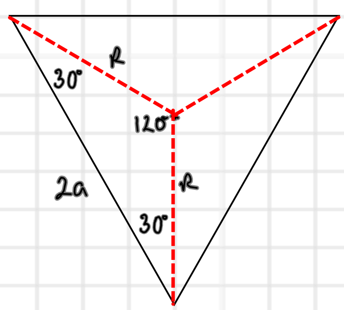
(b) A regular tetrahedron has four faces, each of which is an equilateral triangle.

A wooden puzzle consists of several pieces that can be assembled to make a regular tetrahedron. The manufacturer wants to package the assembled tetrahedron in a clear cylindrical container, with one face flat against the bottom.

If the length of one edge of the tetrahedron is $2a$, show that the volume of the smallest possible cylindrical container is $\left(\dfrac{8\sqrt{6}}{9}\right)\pi a^3$.



Consider base triangle



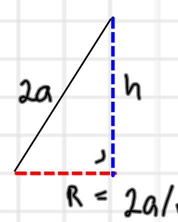
R = Radius of cylinder

Sine Rule

$$\frac{R}{\sin 30^\circ} = \frac{2a}{\sin 120^\circ}$$

$$R = \frac{2a \sin 30^\circ}{\sin 120^\circ} = \frac{2a}{\sqrt{3}}$$

Pythagoras



$R = 2a/\sqrt{3}$

$$h^2 = 4a^2 - 4a^2/3 = 8a^2/3$$

$$h = a\sqrt{8}/\sqrt{3} = a\,2\sqrt{6}/3$$

Volume of a cylinder

$$V = \pi R^2 h$$

$$V = \pi\,(2a/\sqrt{3})^2\,(2a\sqrt{6}/3) = \left[\frac{8\sqrt{6}}{9}\right]\pi a^3$$

Part 3: Algebra

A Cubic function f is defined for $x \in \mathbb{R}$ as

$f: x \to x^3 + (1-k^2)x + k$, where k is a constant

(a) show that -k is a root of f.

If -k is a root then $f(-k) = 0$

$$f(-k) = (-k)^3 + (1-k^2)(-k) + k$$
$$= -k^3 - k + k^3 + k$$
$$= 0$$

(b) find in terms of k the other two roots

Divide by related factor $(x+k)$

$$\begin{array}{r l}
 & x^2 - kx + 1 \\
x+k \,\big)\!\!\! & \cancel{x^3} + 0x^2 + (1-k^2)x + k \\
 & \mp\cancel{x^3} \mp kx^2 \\
 & -\cancel{kx^2} + (1-k^2)x \\
 & \pm\cancel{kx^2} \pm k^2x \\
 & \cancel{x} + \cancel{k} \\
 & \mp\cancel{x} \mp\cancel{k}
\end{array}$$

Factorise

$$x^2 - kx + 1 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{k \pm \sqrt{k^2 - 4}}{2}$$

$$f: x \to x^3 + (1-k^2)x + k\ ,$$

(c) Find the Set of values of k for which f has exactly one real Root.

$\Delta < 0$

⇒ one real root

From part (b)

$$x = \frac{k \pm \sqrt{k^2 - 4}}{2}$$

Δ

$\Rightarrow \quad k^2 - 4 < 0$

$k^2 < 4$

$k < 4 \quad$ OR $\quad k > -4$

$\Rightarrow \quad -4 < k < 4$

(a) Solve the simultaneous equations

$$2x + 8y - 3z = -1 \quad ①$$
$$2x - 3y + 2z = 2 \quad ②$$
$$2x + y + z = 5 \quad ③$$

① - ③

$$\begin{aligned} \cancel{2x} + 8y - 3z &= -1 \\ -\cancel{2x} - y - z &= -5 \\ \hline 7y - 4z &= -6 \quad ④ \end{aligned}$$

③ - ②

$$\begin{aligned} \cancel{2x} + y + z &= 5 \\ -\cancel{2x} + 3y - 2z &= -2 \\ \hline 4y - z &= 3 \quad ⑤ \end{aligned}$$

④ - 4⑤

$$\begin{aligned} 7y - \cancel{4z} &= -6 \\ -16y + \cancel{4z} &= -12 \\ \hline -9y &= -18 \\ y &= 2 \end{aligned}$$

Sub into ⑤

$$4(2) - z = 3$$
$$8 - z = 3$$
$$z = 5$$

Sub into ③

$$2x + 2 + 5 = 5$$
$$2x = -2$$
$$x = -1$$

Check:

① $2(-1) + 8(2) - 3(5) = -1$ ✓

② $2(-1) - 3(2) + 2(5) = 2$ ✓

③ $2(-1) + 2 + 5 = 5$ ✓

(b) Find the set of all real values of x for which $\dfrac{2x-5}{x-3} \le \dfrac{5}{2}$

$$\frac{2x-5}{x-3} - \frac{5}{2} \le 0$$

$$\frac{2(2x-5)-5(x-3)}{2(x-3)} = \frac{4x-10-5x+15}{2(x-3)}$$

$$= \frac{-x+5}{2x-6} \le 0$$

Consider $\dfrac{-x+5}{2x-6} = 0 \Rightarrow -x+5=0 \Rightarrow x=5$

Consider $\dfrac{-x+5}{2x-6} < 0$ For this to be true, denominator and numerator must have different signs.

Case 1: positive numerator negative denominator

$$\left.\begin{aligned} -x+5 > 0 &\Rightarrow -x > -5 \Rightarrow x < 5 \\ 2x-6 < 0 &\Rightarrow 2x < 6 \Rightarrow x < 3 \end{aligned}\right\} \Rightarrow x<3$$

Case 2: negative numerator and positive denominator

$$\left.\begin{aligned} -x+5 < 0 &\Rightarrow -x < -5 \Rightarrow x > 5 \\ 2x-6 > 0 &\Rightarrow 2x > 6 \Rightarrow x > 3 \end{aligned}\right\} \Rightarrow x>5$$

check:

$x>5$ eg.. $x=6 \Rightarrow \dfrac{-(6)+5}{2(6)-6} = \dfrac{-1}{6} < 0$, True ✓

$x<3$ eg $x=0 \Rightarrow \dfrac{-(0)+5}{2(0)-6} = -\dfrac{5}{6} < 0$, True ✓

Solution: $3 > x \le 5,\ x \in \mathbb{R}$