Let $y = \dfrac{1-\cos x}{1+\cos x}$. Show $\dfrac{dy}{dx} = t + t^3$

where $t = \tan\left(\frac{x}{2}\right)$

Quotient Rule

$y = \frac{u}{v}$

$\dfrac{dy}{dx} = \dfrac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$y = \cos x$

$\dfrac{dy}{dx} = -\sin x$

$u = 1 - \cos x$ $\qquad$ $v = 1 + \cos x$

$\dfrac{du}{dx} = \sin x$ $\qquad$ $\dfrac{dv}{dx} = -\sin x$

$$\Rightarrow \frac{dy}{dx} = \frac{(1+\cos x)(\sin x) - (1-\cos x)(-\sin x)}{(1+\cos x)^2}$$

expand

$$= \frac{\sin x + \cancel{\cos x \sin x} + \sin x - \cancel{\cos x \sin x}}{(1+\cos x)^2}$$

$$= \frac{2\sin x}{(1+\cos x)^2}$$

use log tables formula to express these in terms of $\left(\frac{x}{2}\right)$

$$\frac{dy}{dx} = \frac{2\sin x}{(1+\cos x)^2}$$

$\sin 2A = 2\sin A\cos A$

$\Rightarrow \sin x = 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)$

$\cos^2 A = \frac{1}{2}(1+\cos 2A)$

$\Rightarrow 1 + \cos 2A = 2\cos^2 A$

$\Rightarrow 1 + \cos x = 2\cos^2\left(\frac{x}{2}\right)$

$$\Rightarrow \frac{dy}{dx} = \frac{2\left[2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)\right]}{\left[2\cos^2\left(\frac{x}{2}\right)\right]^2}$$

$$= \frac{\cancel{4}\sin\left(\frac{x}{2}\right)\cancel{\cos\left(\frac{x}{2}\right)}}{\cancel{4}\cos^{\cancel{4}\,3}\left(\frac{x}{2}\right)}$$

$$= \frac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)\cos^2\left(\frac{x}{2}\right)}$$

$\dfrac{1}{\cos\theta} = \sec\theta$

$$= \tan\left(\frac{x}{2}\right)\sec^2\left(\frac{x}{2}\right)$$

$\sec^2\theta = 1 + \tan^2\theta$

$$= \tan\left(\frac{x}{2}\right)\left[1 + \tan^2\left(\frac{x}{2}\right)\right]$$

$$= \tan\left(\frac{x}{2}\right) + \tan^3\left(\frac{x}{2}\right)$$

$$= t + t^3$$ QED.