Financial Maths

Section 5.1

Key words

future value	**present value**	**compound interest**	**depreciation**	**annuity**
instalment savings	**instalment payments**	**loans**	**mortgage**	

Geometric series are very important in the world of economics where we meet such terms as (i) Compound Interest (ii) Future Value (iii) Present Value (iv) Pensions (v) Annuity (vi) Mortgage repayments (vii) Instalment savings, etc.

Compound Interest

$$F = P(1 + i)^t$$

P = Principal = Present value

F = Future Value

I = Interest = A - P

i = Rate of interest

t = Time (no. of time intervals on which interest is added)

Example 1

Find the future value of €5000 invested at 4% (AER) per annum, compounded annually, for 6 years. Find also the interest earned over the period.

$P = €5000$
$r = 4\%$
$t = 6$ years

$F = P(1+i)^t$

$F = 5000(1 \cdot 04)^6 = €6{,}326 \cdot 60$

$I = F - P$

$I = 6326 \cdot 60 - 5000 = €1{,}326 \cdot 60$

Example 2

An investment bond offers a return of 15% if invested for 4 years. Calculate the AER (annual equivalent rate) for this bond, correct to two places of decimals.

$i = $ AER

$$\cancel{P}(1+i)^4 = \cancel{P}(1 \cdot 15)^1$$

$$\Rightarrow 1+i = \sqrt[4]{1 \cdot 15}$$

$$i = \sqrt[4]{1 \cdot 15} - 1$$

$$= 0 \cdot 035558$$

$$= 3 \cdot 5558\ \%$$

$$\approx 3 \cdot 56\ \% \quad (2\ \text{d.p.})$$

Example 3

€5000 is invested at 4% AER. If the interest is added monthly, find the future value of this investment after (i) $3\frac{1}{2}$ years (ii) 5 years 2 months.

i = AER
R = MER

$F = ?$
$P = €5000$
$i = 4\%$

FIND MER?

$(1+R)^{12} = (1+i)$

$R = \sqrt[12]{1+i} - 1 = \sqrt[12]{1.04} - 1 = 0.003274$
$= 0.3274\%$

(i) $t = 3\frac{1}{2}$ years $= 42$ months

$F = P(1+i)^t$

$F = 5000(1.003274)^{42} = €5735.77$

(ii) $t =$ 5 years 2 months $= 62$ months

$F = 5000(1.003274)^{62} = €6123.26$

Exercise 5.1

1. Find the future value, correct to 2 places of decimals, of €3000 invested for 10 years at an annual equivalent rate (AER) of 3%.

$P = €3000$
$t = 10$ years
$R = 3\%$

$F = P(1+i)^t$

$F = 3000(1.03)^{10} = €4,031.75$ ✓

2. Given an AER of 2.5%, find the future value, correct to 2 places of decimals, of €5000 invested for 8 years. What interest would be paid on this investment?

$F = ?$
$i = 2.5\%$
$P = €5000$
$t = 8$ years

$F = P(1+i)^t$ $\qquad F = 5000(1.025)^8 = €6{,}091.01$

$I = F - P$ $\qquad 6091 - 5000 = €1091$

3. Given $(1 + r)^{12} = (1 + i)$, where r is the interest rate per month and i the interest rate per year, find r in terms of i.

$(1+r)^{12} = (1+i)$

12th Root both sides $\qquad 1 + r = \sqrt[12]{1+i}$

-1 $\qquad r = \sqrt[12]{1+i} - 1$

* this Result should be memorised!

4. What monthly rate of interest, correct to 2 places of decimals, is equivalent to an annual rate of (i) 6% (ii) 2.5% (iii) 4% ?

from Q.3

$$r = \sqrt[12]{1+i} - 1$$

$r =$ MER
$i =$ AER

(i) $r = \sqrt[12]{1.06} - 1$

decimal $= 4.868 \times 10^{-3} = 0.004868$

% $\approx 0.49\%$ (2 d.p.)

(ii) $r = \sqrt[12]{1.025} - 1$

decimal $= 2.0598 \times 10^{-3} = 0.002598$

% $\approx 0.26\%$

(iii) $r = \sqrt[12]{1.04} - 1$

decimal $= 3.2737 \times 10^{-3} = 0.0032737$

% $\approx 0.33\%$

5. Sean invested €4500 for five years in EUROBANK.
His investment amounted to €5607.82 at the end of its term.
Find the AER that applied to his investment.

$P =$ € 4500
$F =$ € 5607.82
$t = 5$ years
$i = ?$

$F = P(1+i)^t$

$$5607.82 = 4500(1+i)^5$$

$$\frac{5607.82}{4500} = (1+i)^5$$

$$\sqrt[5]{\frac{5607.82}{4500}} = 1+i$$

$i = \sqrt[t]{\frac{F}{P}} - 1$

$$\Rightarrow i = \sqrt[5]{\frac{5607.82}{4500}} - 1$$

$$i = 0.045 = 4.5\%$$

6. Sandra wins €15 000 in a draw and invests it in a credit union where the AER is 3.5%.
Copy and complete this chart, showing how the value of her money changes over the five years of the investment.

Year	Principal	Interest €
One	€15 000	525
Two	€ 15 525	1 068·38
Three	€ 16 068·38	1630·77
Four	€ 16 630·77	2 212·85
Five	€ 17 212·85	2 815·30

$F = P(1+i)^t$

$I = F - P$

Yr.1 $F_1 = 15000(1\cdot035)^1 = 15,525$

$I_1 = 15525 - 15000 = 525$

Yr.2 $F_2 = 15,525(1\cdot035)^1 = €16,068\cdot38$

$I_2 = 16,068\cdot38 - 15000 = 1068\cdot38$

Yr.3 $F_3 = 16,068\cdot38\,(1\cdot035)^1 = €16,630\cdot77$

$I_3 = 16630\cdot77 - 15000 = 1630\cdot77$

Yr.4 $F_4 = 16,630\cdot77\,(1\cdot035)^1 = €17,212\cdot85$

$I_4 = 17212\cdot85 - 15000 = 2212\cdot85$

Yr.5 $F_5 = 17,212\cdot85\,(1\cdot035)^1 = €\,17,815\cdot30$

$I_5 = 17815\cdot30 - 15000 = 2815\cdot30$

7. Kamil asks for interest to be added half-yearly to his account.
If the bank offers an AER of 4%, find, correct to four significant figures, the equivalent half-yearly rate.

i = AER

R = BER

4% AER = ? Twice yearly Rate

? BER (Bi-annual E.R.)

$(1+R)^2 = (1+i)$

$\Rightarrow R = \sqrt[2]{1+i} - 1$

$R = \sqrt{1\cdot04} - 1$

$= 0\cdot0198$

$= 1\cdot98\%$

8. Find the future value of €6500 invested for 6 years 4 months if the monthly equivalent rate is 1.932%.

$F = P(1+i)^t$

$P = €\,6500$

$i = 1.932\% = 0.01932$

$t = 6$ years 4 months $= 76$ months

$F = ?$

$F = 6500\,(1.01932)^{76}$

$= €\,27830.10$

9. €12 000 is invested at an AER of 3.5%.
Find the value of the investment after
(i) 5 years 3 months (ii) 8 years 2 months (iii) 10 years 6 months.

Let $i =$ AER , $R =$ MER

$R = ?$

$(1+i)^1 = (1+R)^{12}$

$R = \sqrt[12]{1+i} - 1 = \sqrt[12]{1.035} - 1 = 0.00287$

$= 0.287\,\%$

$F = P(1+i)^t$

(i) $t = 5$ years 3 months $= 63$ months

$F = 12000\,(1.00287)^{63} = €\,14{,}374.53$

(ii) $t = 8$ years 2 months $= 98$ months

$F = 12000\,(1.00287)^{96} = €\,15800.35$

(iii) $t = 10$ years 6 months $= 126$ months

$F = 12000\,(1.00287)^{126} = €\,17218.92$

Example 4

The local GAA club runs a draw.
You win first prize and you are offered

(a) €15 000 now **or**
(b) €18 000 in four years time.

Which prize should you choose to have the greatest value? Assume a discount rate of 4%.

When calculating present value, the rate *i*% is often referred to as the "*discount rate*".

Method 1

$F = P(1+i)^t$

Evaluate Future Value of €15000 @ 4% AER in 4 years time?

$F = 15000(1.04)^4 = €17,547.88$

comment: this is less than €18,000 in 4 years.

method 2

$P = \frac{F}{(1+i)^t}$

Evaluate Present Value of what will have a value of €18,000 in 4 years @ AER 4%.

$P = \frac{18000}{(1.04)^4} = €15,386$

Comment: this is more than €15,000 now.

Example 5

In how many years would €5000 increase in value to €6500 if invested at an AER of 3.5%?

$F = P(1+i)^t$

$F = 6500$

$P = 5000$

$R = 3.5\%$

$t = ?$

$\frac{F}{P} = (1+i)^t$

$\Rightarrow \frac{6500}{5000} = (1.035)^t$

$1.3 = 1.035^t$

If $a = b^n$

$n = \log_b a$

$t = \log_{1.035} 1.3 = 7.626$ years

10. If a bank offers a discount rate of 4.2%, find the present value of €10 000 due to be paid in 10 years time.

$F = P(1+i)^t$

$\Rightarrow P = \dfrac{F}{(1+\ )^t}$

$F = 10\,000 \quad R = 4.2\% \quad t = 10$

$$P = \frac{10000}{(1.042)^{10}} = €\ 6{,}627.09$$

11. Jonathan is 12 years old. When he is 21, he is due to inherit €25 000. What is the present value of his inheritance assuming a discount rate of 4.5%?

$P = \dfrac{F}{(1+i)^t}$

$t = 21 - 12 = 9$ years

$F = €\ 25{,}000$

$i = 4.5\%$

$P = ?$

$$P = \frac{25000}{(1.045)^9} = €\ 16{,}822.61$$

12. €50000 is invested in a bank offering an AER of 3.5%.
How long will it take this investment to double in value?

$F = P(1+i)^t$

$P = €50{,}000$
$F = 2(50000) = €100{,}000$
$i = 3.5\%$
$t = ?$

$\frac{F}{P} = (1+i)^t$

$\frac{100\,000}{50\,000} = 1.035^t$

$2 = 1.035^t$

If $a = b^n$

$\Rightarrow \log_b a = n$

$t = \log_{1.035} 2 \approx 20$ years

* note the formula for t is always

$t = \log_{(1+i)}\left(\frac{F}{P}\right)$

13. I plan to borrow €175 000 to buy a house.
If the bank charges an AER of 4.5%, what would this loan amount to in 20 years, assuming no repayments?

$F = P(1+i)^t$

$\left.\begin{array}{l} P = 175\,000 \\ R = 4.5\% \\ t = 20 \text{ years} \end{array}\right\} F = ?$

$F = 175\,000(1.045)^{20}$

$= €422{,}049.95$

14. Using ~~(a) trial and error and~~ (b) logs, find how many years it will take €1130 to have a future value of €3000 if invested at 5% per annum compound interest.

$F = P(1+i)^t$

$\Rightarrow (1+i)^t = \frac{F}{P}$

If $a = b^n$

$\Rightarrow \log_b a = n$

$$\left.\begin{aligned} A &= 3000 \\ P &= 1130 \\ R &= 5\% \end{aligned}\right\} \quad t = ?$$

$$1.05^t = \frac{3000}{1130}$$

$$1.05^t = 2.655$$

$$t = \log_{1.05} 2.655 = 20 \text{ years}$$

15. The formula $(1 + r)^{12} = (1 + i)$, where r is the interest rate per month and i the interest rate per annum, is used to calculate the effective monthly interest rate.

(i) If 6% interest is offered per year, calculate the effective monthly rate correct to four places of decimals.

(ii) If r was simply calculated by dividing the yearly interest by 12, calculate using both methods, the difference in future values of €10 000 in 3 years at 6% per annum, if the interest is compounded monthly.

(iii) What is the minimum number of places of decimals that need to be taken in calculating r before a difference is noted in future values?

(i) $R = \text{MER}$, $i = \text{AER}$

$(1+R)^{12} = (1+i)$

$$R = \sqrt[12]{1+i} - 1 = \sqrt[12]{1.06} - 1 = 0.0048755$$
$$= 0.4876\%$$

(ii) If $^*R = \frac{i}{12}$

$$^*R = \frac{i}{12} = \frac{0.06}{12} = 0.005 = 0.5\%$$

Difference in future values?

$^*R$ method 2: $A = 10000(1.005)^{36} = €11,966.81$

R method 1: $A = 10000(1.004867)^{36} = €11,909.93$

difference $= €56.88$

(iii) correct to 2 d.p. $0.4876\% = 0.49\%$ which will give a different F than 0.5%

16. Anna invests €15 000 at an AER of 3%. After two years, she withdraws €2000 but leaves the remainder of her investment for a further three years.
What is the value of her investment at the end of this period?

$F = P(1+i)^t$

Separate question into "years 1 and 2" and "years 3 to 5".

Years 1 & 2

$P =$ €15,000
$i = 3\%$
$t = 2$ years
$F = ?$

$F = 15000(1.03)^2 =$ €15,913.50

Years 3 to 5

$P = 15913.50 - 2000 =$ €13913.50
$i = 3\%$
$t = 3$ years
$F = ?$

$F = 13913.50(1.03)^3 =$ €15,203.66