Geometry — What do you need to know?

Constructions:

1.* Bisector of a given angle.

2.* Perpendicular bisector of a segment.

3.* Line perpendicular to a given line, l, through a point not on l.

4.* Line perpendicular to a given line, l, through a point on l.

5.* Line parallel to a given line through a given point.

6.* Division of a segment into two or three equal segments.

7.* Division of a segment into any number of equal segments.

8.* Line segment of given length on given ray.

9.* Angle of given number of degrees with a given ray as one arm.

10.* Triangle, given lengths of three sides.

11.* Triangle, given SAS data.

12.* Triangle, given ASA data.

13.* Right-angled triangle, given length of hypotenuse and one other side.

14.* Right-angled triangle, given length of hypotenuse and one other angle.

15.* Rectangle, given side lengths.

16. Circumcentre and circumcircle of a given triangle.

17. Incentre and incircle of given triangle.

18. Angle of $60°$, without protractor or set square.

19. Tangent to a given circle at a point on the circle.

20. Parallelogram given the lengths of the sides and the measure of the angles.

21. Centroid of a triangle.

22.* Orthocentre of triangle.

Theorems:

Theorem 11 *If three parallel lines cut off equal segments on some transversal line, then they will cut off equal segments on any other transversal.*

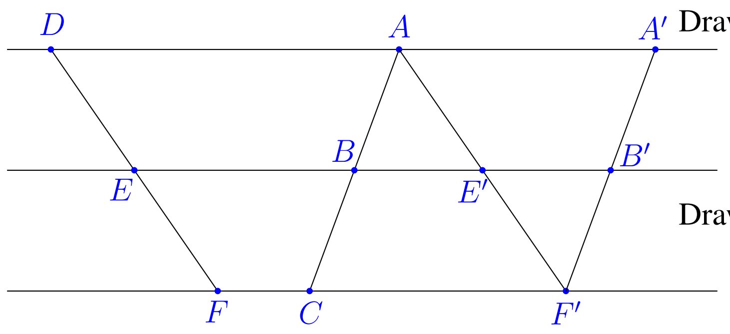
Theorem 12 *Let $\triangle ABC$ be a triangle. If a line l is parallel to BC and cuts $[AB]$ in the ratio $s : t$, then it also cuts $[AC]$ in the same ratio.*

Theorem 13 *If two triangles $\triangle ABC$ and $\triangle A'B'C'$ are similar, then their sides are proportional, in order:*

$$\frac{|AB|}{|A'B'|} = \frac{|BC|}{|B'C'|} = \frac{|CA|}{|C'A'|}$$

Theorems

Theorem 11 *If three parallel lines cut off equal segments on some transversal line, then they will cut off equal segments on any other transversal*
Suppose $AD \parallel BE \parallel CF$ and $|AB| = |BC|$. Prove that $|DE| = |EF|$.



Draw $AE' \parallel DE$

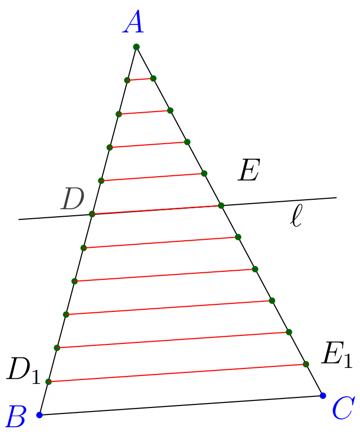
Draw $F'B' \parallel AB$

$\|B'F'\|$	$=$	$\|BC\|$	Theorem 9
	$=$	$\|AB\|$	by Assumption
$\|\angle BAE'\|$	$=$	$\|\angle E'F'B'\|$	Alternate angles
$\|\angle AE'B\|$	$=$	$\|\angle F'E'B'\|$	Vertically Opposite Angles
$\Rightarrow \triangle ABE'$	is congruent to	$\triangle F'B'E'$	ASA
$\Rightarrow \|AE'\|$	$=$	$\|F'E'\|$	

But

$$|AE'| = |DE| \text{ and } |F'E'| = |FE| \text{ by Theorem 9}$$

Hence $|DE| = |EF|$
Theorem 12 *Let $\triangle ABC$ be a triangle. If a line l is parallel to BC and cuts $[AB]$ in the ratio $m : n$, then it also cuts $[AC]$ in the same ratio*



Let ℓ cut $[AB]$ in D in the ratio $m : n$ with natural numbers m and n. Thus there are points

$$D_0 = B, D_1, D_2, \ldots, D_{m-1}, D_m = D, D_{m+1}, \ldots, D_{m+n-1}, D_{m+n} = A$$

equally spaced along $[BA]$. This means that

$$|D_0D_1| = |D_1D_2| = \cdots |D_{m+n-1}D_{m+n}|$$

Draw lines $D_1E_1, D_2E_2, \ldots$ par allel to BC with $E_1, E_2 \ldots$ on $[AC]$.

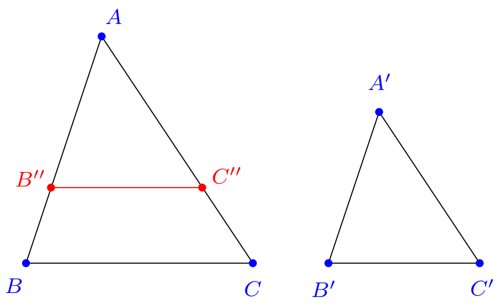
Then all the segments

$$[CE_1], [E_1E_2], [E_2E_3], \ldots, [E_{m+n-1}A]$$

have the same length, by Theorem 11.
The Axiom of Parallels tells us that $E_m = E$ is the point where ℓ cuts $[AC]$.
Hence E divides $[CA]$ in the ratio $m : n$

Theorem 13 *If two triangles $\triangle ABC$ and $\triangle A'B'C'$ are similar, then their sides are proportional, in order*:

$$\frac{|AB|}{|A'B'|} = \frac{|BC|}{|B'C'|} = \frac{|CA|}{|C'A'|}$$



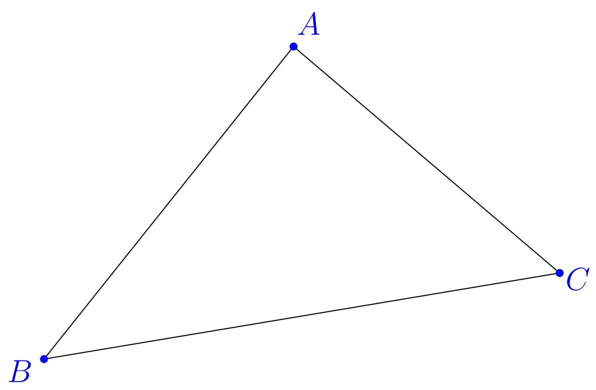
Suppose that $|A'B'| \leq |AB|$. Pick B'' on $[AB]$ with $|AB''| = |A'B'|$, and C'' on $[AC]$ with $|AC''| = |A'C'|$. Then

$$\begin{array}{rcll}
\triangle AB''C'' & \text{is congruent to} & \triangle A'B'C' & \text{[SAS]} \\
\Rightarrow |\angle AB''C''| & = & |\angle ABC & \\
\Rightarrow B''C'' & \parallel & BC & \text{Corresponding Angles} \\
\Rightarrow \frac{|A'B'|}{|A'C'|} & = & \frac{|AB''|}{|AC''|} & \text{Choice of } B'', C'' \\
 & = & \frac{|AB|}{|AC|} & \text{Theorem 12} \\
\Rightarrow \frac{|AC|}{|A'C'|} & = & \frac{|AB|}{|A'B'|} &
\end{array}$$

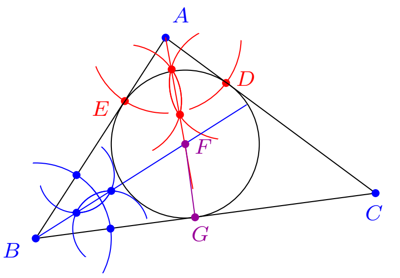
Similarly $\dfrac{|BC|}{|B'C'|} = \dfrac{|AB|}{|A'B'|}$

Example — Leaving Certificate 2010 Q3

(a) Construct the incircle of the triangle ABC below using only a compass and straight edge. Show all construction lines clearly.



Solution



With the compass at A mark equal segments, $[AD]$ and $[AE]$

With the compass at E and D draw equal arcs

Draw the bisector of $\angle BAC$

Similarly, bisect $\angle CBA$

Find F, the point of intersection of the angle bisectors

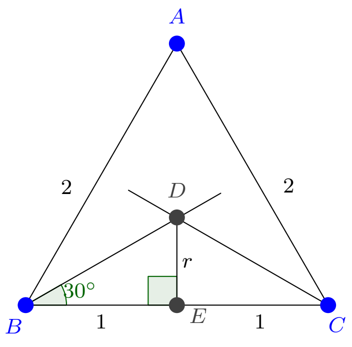
Draw $[FG] \perp [BC]$

With F as centre draw a circle with radius $|FG|$

Example — Leaving Certificate 2010 Q3

(b) An equilateral triangle has sides of length 2 units.
Find the area of its incircle.

Solution



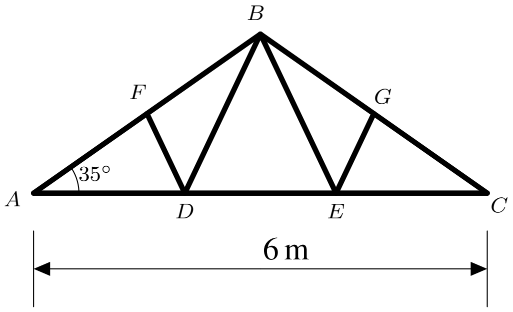
Since $\triangle ABC$ is equilateral the incentre and the circumcentre coincide. Hence, DE bisects $[BC]$.

$$\begin{aligned} \tan 30^\circ &= \frac{r}{1} \\ &= \frac{1}{\sqrt{3}} \\ \Rightarrow r &= \frac{1}{\sqrt{3}} \\ A &= \pi r^2 \\ &= \frac{\pi}{3} \end{aligned}$$

Example — Leaving Certificate 2010 Q9(b)

Roofs of buildings are often supported by frameworks of timber called *roof trusses*.

A quantity surveyor needs to find the total length of timber needed in order to make the triangular truss shown below.

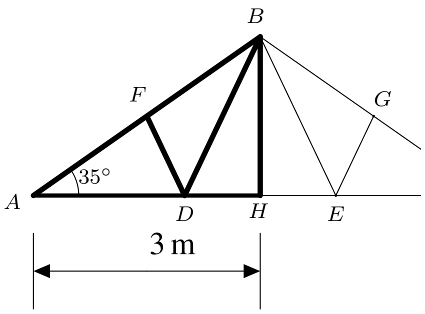


The length of [AC] is 6 metres, and the pitch of the roof is 35°, as shown.
$|AD| = |DE| = |EC|$ and $|AF| = |FB| = |BG| = |GC|$.

(i) Calculate the length of $[AB]$, in metres, correct to two decimal places.

Solution

$$\begin{aligned} |AH| &= 3\,\text{m} \\ \cos 35^\circ &= \frac{3}{|AB|} \\ \Rightarrow |AB| &= \frac{3}{\cos 35^\circ} \\ &\approx 3.66232 \\ \Rightarrow |AB| &= 3.66\,\text{m (to 2 decimal places)} \end{aligned}$$



(ii) Calculate the total length of timber required to make the truss.

Solution

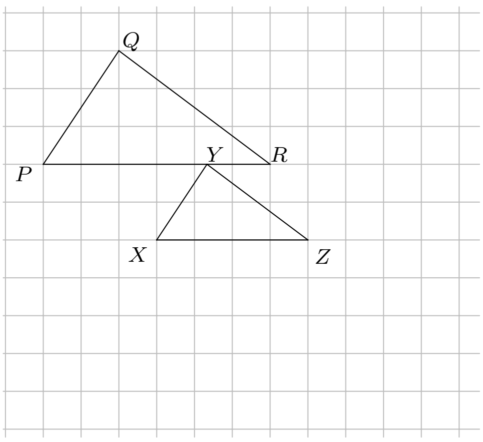
$$\begin{aligned} |FD|^2 &= 1.83^2 + 2^2 - 2 \times 1.83 \times 2 \times \cos 35^\circ \\ &= 1.352707 \\ \Rightarrow |FD| &= 1.163\,\text{m} \end{aligned}$$

$$\begin{aligned} |BD|^2 &= 2^2 + 3.66^2 - 2 \times 2 \times 3.66 \times \cos 35^\circ \\ &= 5.403214 \\ \Rightarrow |BD| &= 2.325\,\text{m} \end{aligned}$$

$$\begin{aligned} \text{Total length} &= 6 + 2 \times 3.662 + 2 \times 1.163 + 2 \times 2.325 \\ &= 20.3\,\text{m} \end{aligned}$$

Example — 2011 Question 4

Two triangles are drawn on a square grid as shown. The points P, Q, R, X and Z are on vertices of the grid, and the point Y lies on $[PR]$. The triangle PQR is an enlargement of the triangle XYZ.



(a) Calculate the scale factor of the enlargement, showing your work.

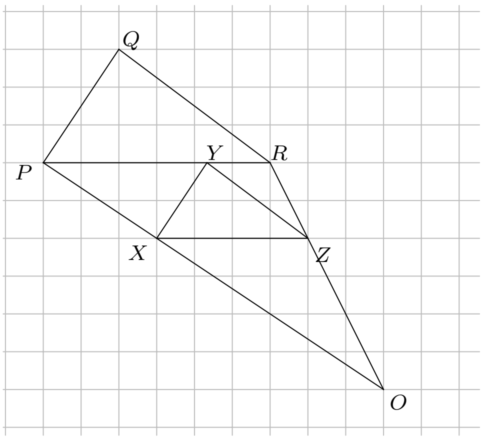
(b) By construction, or otherwise, locate the centre of the enlargement on the diagram above.

(c) Calculate $|YR|$ in grid units.

Solution

(a) Calculate the scale factor of the enlargement, showing your work.

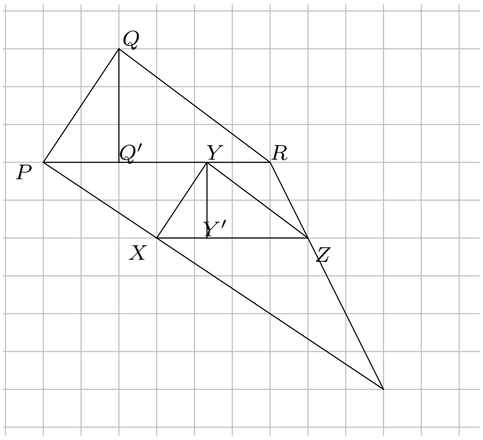
$$\begin{aligned} \frac{|PR|}{|XZ|} &= \frac{6}{4} \\ &= \frac{3}{2} \end{aligned}$$



(b) See diagram.

(c) Calculate $|YR|$ in grid units.

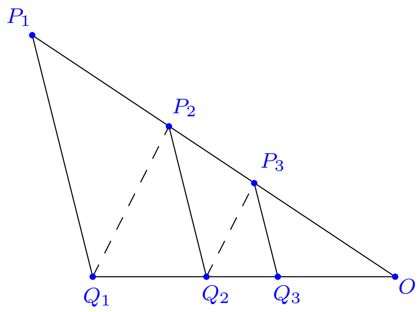
$$\begin{aligned} |Y'Z| &= \frac{2}{3}|Q'R| \\ &= \frac{2}{3} \times 4 \\ &= \frac{8}{3} \\ \Rightarrow |YR| &= \frac{8}{3} - 1 \\ &= \frac{5}{3} \end{aligned}$$



Example — 2011 Question 6

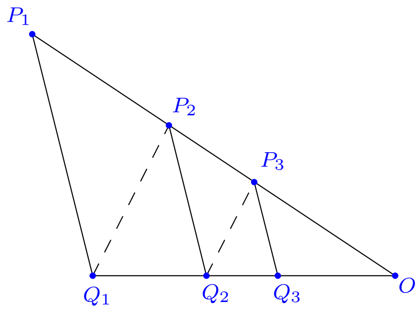
In the diagram, P_1Q_1, P_2Q_2 and P_3Q_3 are parallel and so also are Q_1P_2 and Q_2P_3.
Prove that

$$|P_1Q_1| \times |P_3Q_3| = |P_2Q_2|^2$$



Solution

$$\begin{aligned}
\frac{|OP_3|}{|OP_2|} &= \frac{|OQ_2|}{|OQ_1|} \quad (P_3Q_2 \parallel P_2Q_1)\\
\frac{|OP_2|}{|OP_1|} &= \frac{|OQ_2|}{|OQ_1|} \quad (P_2Q_2 \parallel P_1Q_1)\\
\Rightarrow \frac{|OP_3|}{|OP_2|} &= \frac{|OP_2|}{|OP_1|}\\
\frac{|OP_2|}{|OP_1|} &= \frac{|P_2Q_2|}{|P_1Q_1|} \quad \text{(Similar triangles)}\\
\frac{|OP_3|}{|OP_2|} &= \frac{|P_3Q_3|}{|P_2Q_2|} \quad \text{(Similar triangles)}\\
\Rightarrow \frac{|P_2Q_2|}{|P_1Q_1|} &= \frac{|P_3Q_3|}{|P_2Q_2|}\\
\Rightarrow |P_1Q_1| \times |P_3Q_3| &= |P_2Q_2|^2
\end{aligned}$$

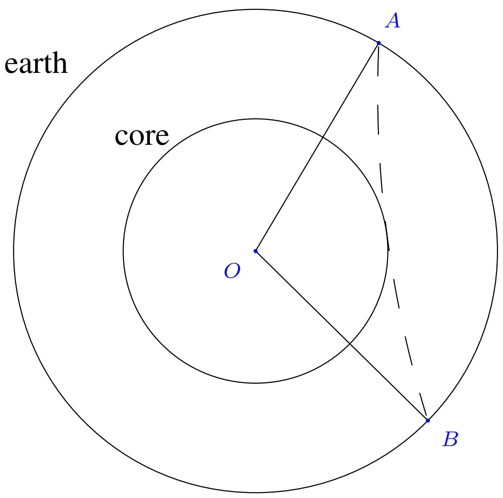


Example — 2011 Question 7(c)

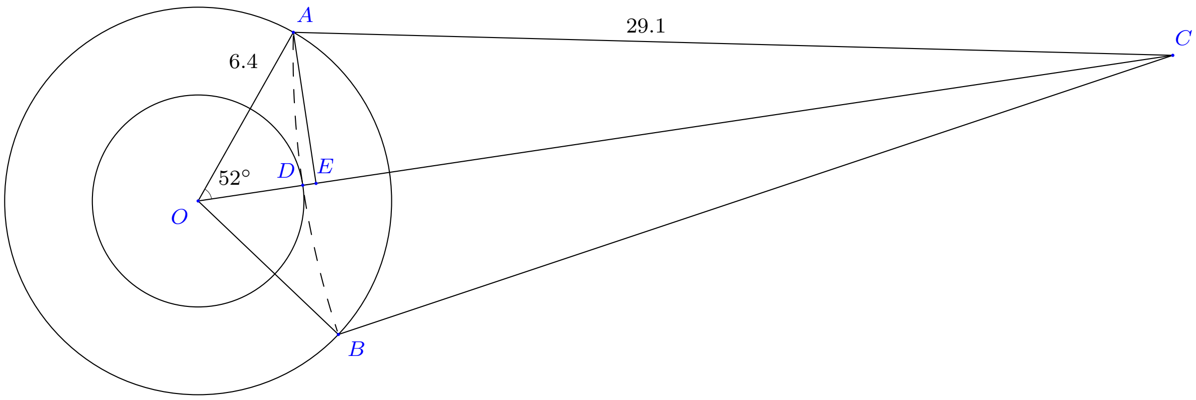
Scientists use information about seismic waves from earthquakes to find out about the internal structure of the earth.

The diagram below represents a circular cross-section of the earth. The dashed curve represents the path of a seismic wave travelling through the earth from an earthquake near the surface at A to a monitoring station at B. The radius of the earth is 6.4 units and the path of this wave is a circular arc of radius 29.1 units, where 1 unit = 1000 km. Based on information from other stations, it is known that this particular path just touches the earths core. The angle AOB measures $104°$, where O is the centre of the earth.

Find the radius of the earth's core.



Solution



$$\begin{aligned}|AE| &= 6.4\sin 52^\circ \\ &= 5.04\end{aligned}$$

$$\begin{aligned}|OE| &= 6.4\cos 52^\circ \\ &= 3.94\end{aligned}$$

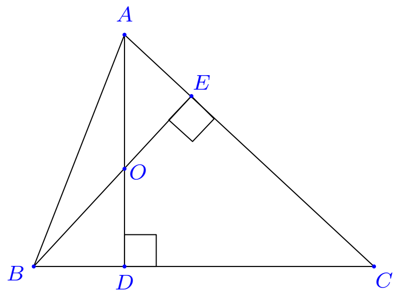
$$\begin{aligned}|CE| &= \sqrt{29.1^2 - 5.04^2} \\ &= 28.66\end{aligned}$$

$$\begin{aligned}|DE| &= 29.1 - 28.66 \\ &= 0.44\end{aligned}$$

$$\begin{aligned}|OD| &= 3.94 - 0.44 \\ &= 3.5 \text{ units (or 3500 km)}\end{aligned}$$

Example — Project Maths Sample Paper 2012 Question 6

ABC is a triangle.
D is the point on BC such that $AD \perp BC$.
E is the point on AC such that $BE \perp AC$.
AD and BE intersect at O.
Prove that $|\angle DOC| = |\angle DEC|$.



Solution

$$\begin{aligned} |\angle ADC| &= 90^\circ \\ |\angle OEC| &= 90^\circ \\ \Rightarrow |\angle ADC| + |\angle OEC| &= 180^\circ \end{aligned}$$

$\Rightarrow DOEC$ is a cyclic quadrilateral.

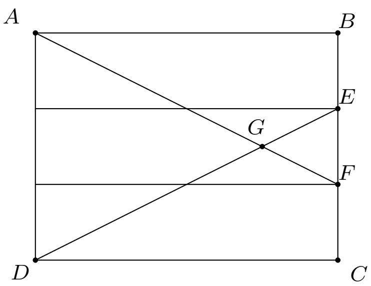
$[DC]$ is a chord of a circle passing through the points D, O, E and C.

$\Rightarrow \angle DOC$ and $\angle DEC$ are angles on the same arc

$\Rightarrow |\angle DOC| = |\angle DEC|$.

Example — Project Maths Sample Paper 2012 Question 8

Anne is having a new front gate made and has decided on the design below.



The gate is 2 metres wide and 1.5 metres high. The horizontal bars are 0.5 metres apart.

(i) Calculate the common length of the bars $[AF]$ and $[DE]$, in metres, correct to three decimal places.

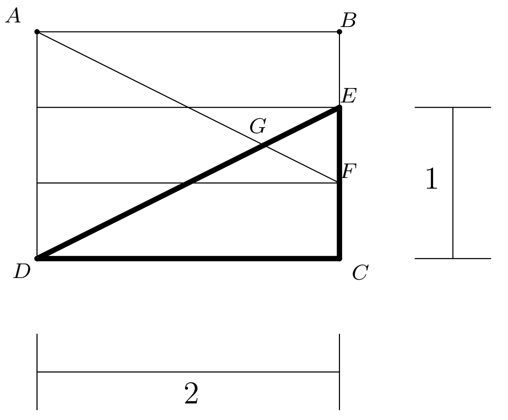
(ii) In order to secure the bar $[AF]$ to $[DE]$, the manufacturer needs to know

- the measure of the angle EGF, and
- the common distance $|AG| = |DG|$

Find these measures. Give the angle correct to the nearest degree and the length correct to three decimal places.

Solution

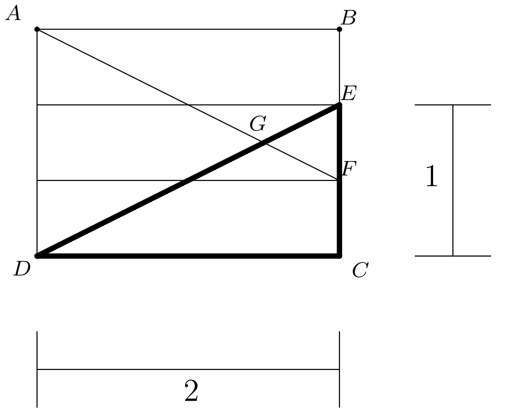
(i) Calculate the common length of the bars $[AF]$ and $[DE]$, in metres, correct to three decimal places.



$$\begin{aligned} |DE|^2 &= 2^2 + 1^2 \\ &= 5 \\ \Rightarrow |DE| &= \sqrt{5} \\ &= 2.236067 \\ &\approx 2.236 \\ &= |AF| \end{aligned}$$

(ii) In order to secure the bar $[AF]$ to $[DE]$, the manufacturer needs to know

- the measure of the angle EGF



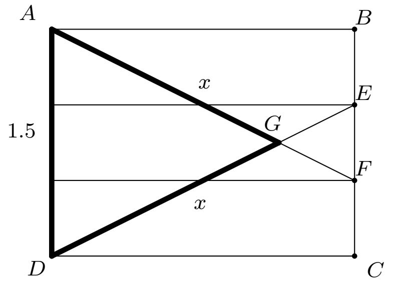
$$\begin{aligned} \tan \angle DEC &= 2 \\ \Rightarrow \angle DEC &= 63.4349^\circ \\ &= \angle AFB \end{aligned}$$

$$\begin{aligned} \Rightarrow \angle EGF &= 180^\circ - 126.8698^\circ \\ &= 53.1302^\circ \\ &= 53^\circ \text{ to the nearest degree} \end{aligned}$$

(ii) In order to secure the bar $[AF]$ to $[DE]$, the manufacturer needs to know

- the common distance $|AG| = |DG|$

Let $|AG| = |DG| = x$, then



$$\begin{aligned} |AD|^2 &= x^2 + x^2 - 2x^2 \cos 53^\circ \\ \Rightarrow 4 &= 2x^2 - 1.20363x^2 \\ 0.79637x^2 &= 4 \\ \Rightarrow x^2 &= 5.02279 \\ \Rightarrow |AG| = |DG| &= 2.241 \text{ to three decimal places} \end{aligned}$$