Geometry for Post-primary School Mathematics

September 2011

1 Introduction

The Junior Certificate and Leaving Certificate mathematics course committees of the National Council for Curriculum and Assessment (NCCA) accepted the recommendation contained in the paper [4] to base the logical structure of post-primary school geometry on the level 1 account in Professor Barry's book [1].

To quote from [4]: We distinguish three levels:

Level 1: The fully-rigorous level, likely to be intelligible only to professional mathematicians and advanced third- and fourth-level students.

Level 2: The semiformal level, suitable for digestion by many students from (roughly) the age of 14 and upwards.

Level 3: The informal level, suitable for younger children.

This document sets out the agreed geometry for post-primary schools. It was prepared by a working group of the NCCA course committees for mathematics and, following minor amendments, was adopted by both committees for inclusion in the syllabus documents. Readers should refer to Strand 2 of the syllabus documents for Junior Certificate and Leaving Certificate mathematics for the range and depth of material to be studied at the different levels. A summary of these is given in sections 9–13 of this document.

The preparation and presentation of this document was undertaken principally by Anthony O'Farrell, with assistance from Ian Short. Helpful criticism from Stefan Bechluft-Sachs, Ann O'Shea, Richard Watson and Stephen Buckley is also acknowledged.

2 The system of geometry used for the purposes of formal proofs

In the following, Geometry refers to plane geometry.

There are many formal presentations of geometry in existence, each with its own set of axioms and primitive concepts. What constitutes a valid proof in the context of one system might therefore not be valid in the context of another. Given that students will be expected to present formal proofs in the examinations, it is therefore necessary to specify the system of geometry that is to form the context for such proofs.

The formal underpinning for the system of geometry on the Junior and Leaving Certificate courses is that described by Prof. Patrick D. Barry in [1]. A properly formal presentation of such a system has the serious disadvantage that it is not readily accessible to students at this level. Accordingly, what is presented below is a necessarily simplified version that treats many concepts far more loosely than a truly formal presentation would demand. Any readers who wish to rectify this deficiency are referred to [1] for a proper scholarly treatment of the material.

Barry's system has the primitive undefined terms **plane, point, line, $<_l$ (precedes on a line), (open) half-plane, distance, and degree-measure**, and seven axioms: A_1: about incidence, A_2: about order on lines, A_3: about how lines separate the plane, A_4: about distance, A_5: about degree measure, A_6: about congruence of triangles, A_7: about parallels.

3 Guiding Principles

In constructing a level 2 account, we respect the principles about the relationship between the levels laid down in [4, Section 2].

The choice of material to study should be guided by applications (inside and outside Mathematics proper).

The most important reason to study synthetic geometry is to prepare the ground logically for the development of trigonometry, coordinate geometry, and vectors, which in turn have myriad applications.

We aim to keep the account as simple as possible.

We also take it as desirable that the official Irish syllabus should avoid imposing terminology that is nonstandard in international practice, or is used in a nonstandard way.

No proof should be allowed at level 2 that cannot be expanded to a complete rigorous proof at level 1, or that uses axioms or theorems that come later in the logical sequence. We aim to supply adequate proofs for all the theorems, but do not propose that only those proofs will be acceptable. It should be open to teachers and students to think about other ways to prove the results, provided they are correct and fit within the logical framework. Indeed, such activity is to be encouraged. Naturally, teachers and students will need some assurance that such variant proofs will be acceptable if presented in examination. We suggest that the discoverer of a new proof should discuss it with students and colleagues, and (if in any doubt) should refer it to the National Council for Curriculum and Assessment and/or the State Examinations Commission.

It may be helpful to note the following non-exhaustive list of salient differences between Barry's treatment and our less formal presentation.

- Whereas we may use set notation and we expect students to understand the conceptualisation of geometry in terms of sets, we more often use the language that is common when discussing geometry informally, such as "the point is/lies on the line", "the line passes through the point", etc.

- We accept and use a much lesser degree of precision in language and notation (as is apparent from some of the other items on this list).

- We state five explicit axioms, employing more informal language than Barry's, and we do not explicitly state axioms corresponding to Axioms A2 and A3 – instead we make statements without fanfare in the text.

- We accept a much looser understanding of what constitutes an **angle**, making no reference to angle-supports. We do not define the term angle. We mention reflex angles from the beginning (but make no use of them until we come to angles in circles), and quietly assume (when the time comes) that axioms that are presented by Barry in the context of wedge-angles apply also in the naturally corresponding way to reflex angles.

- When naming an angle, it is always assumed that the non-reflex angle is being referred to, unless the word "reflex" precedes or follows.

- We make no reference to results such as Pasch's property and the "crossbar theorem". (That is, we do not expect students to consider the necessity to prove such results or to have them given as axioms.)

- We refer to "the number of degrees" in an angle, whereas Barry treats this more correctly as "the degree-measure" of an angle.

- We take it that the definitions of parallelism, perpendicularity and "sidedness" are readily extended from lines to half-lines and line segments. (Hence, for example, we may refer to the opposite sides of a particular quadrilateral as being parallel, meaning that the lines of which they are subsets are parallel).

- We do not refer explicitly to triangles being **congruent** "under the correspondence $(A, B, C) \rightarrow (D, E, F)$", taking it instead that the correspondence is the one implied by the order in which the vertices are listed. That is, when we say "ΔABC is congruent to ΔDEF" we mean, using Barry's terminology, "Triangle [A,B,C] is congruent to triangle [D,E,F] under the correspondence $(A, B, C) \rightarrow (D, E, F)$".

- We do not always retain the distinction in language between an angle and its measure, relying frequently instead on the context to make the meaning clear. However, we continue the practice of distinguishing notationally between the angle $\angle ABC$ and the number $|\angle ABC|$ of degrees in the angle[1]. In the same spirit, we may refer to two angles being equal, or one being equal to the sum of two others, (when we should more precisely say that the two are equal in measure, or that the measure of one is equal to the sum of the measures of the other two). Similarly, with length, we may loosely say, for example: "opposite sides of a parallelogram are equal", or refer to "a circle of radius r". Where ambiguity does not arise, we may refer to angles using a single letter. That is, for example, if a diagram includes only two rays or segments from the point A, then the angle concerned may be referred to as $\angle A$.

Having pointed out these differences, it is perhaps worth mentioning some significant structural aspects of Barry's geometry that are retained in our less formal version:

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[1]In practice, the examiners do not penalise students who leave out the bars.

- The primitive terms are almost the same, subject to the fact that their properties are conceived less formally. We treat **angle** as an extra undefined term.

- We assume that results are established in the same order as in Barry [1], up to minor local rearrangement. The exception to this is that we state all the axioms as soon as they are useful, and we bring the theorem on the angle-sum in a triangle forward to the earliest possible point (short of making it an axiom). This simplifies the proofs of a few theorems, at the expense of making it easy to see which results are theorems of so-called Neutral Geometry[2].

- **Area** is not taken to be a primitive term or a given property of regions. Rather, it is defined for triangles following the establishment of the requisite result that the products of the lengths of the sides of a triangle with their corresponding altitudes are equal, and then extended to convex quadrilaterals.

- **Isometries or other transformations** are not taken as primitive. Indeed, in our case, the treatment does not extend as far as defining them. Thus they can play no role in our proofs.

4   Outline of the Level 2 Account

We present the account by outlining:

1. A list (Section 5), of the terminology for the geometrical concepts. Each term in a theory is either undefined or defined, or at least definable. There have to be some undefined terms. (In textbooks, the undefined terms will be introduced by descriptions, and some of the defined terms will be given explicit definitions, in language appropriate to the level. We assume that previous level 3 work will have laid a foundation that will allow students to understand the undefined terms. We do not give the explicit definitions of all the definable terms. Instead we rely on the student's ordinary language, supplemented sometimes by informal remarks. For instance, we do not write out in cold blood the definition of the **side opposite** a given angle in a triangle, or the

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[2] Geometry without the axiom of parallels. This is not a concern in secondary school.

definition (in terms of set membership) of what it means to say that a line **passes through** a given point. The reason why some terms **must** be given explicit definitions is that there are alternatives, and the definition specifies the starting point; the alternative descriptions of the term are then obtained as theorems.

2. A logical account (Section 6) of the synthetic geometry theory. All the material through to LC higher is presented. The individual syllabuses will identify the relevant content by referencing it by number (e.g. Theorems 1,2, 9).

3. The geometrical constructions (Section 7) that will be studied. Again, the individual syllabuses will refer to the items on this list by number when specifying what is to be studied.

4. Some guidance on teaching (Section 8).

5. Syllabus entries for each of JC-OL, JC-HL, LC-FL, LC-OL, LC-HL.

5 Terms

Undefined Terms: angle, degree, length, line, plane, point, ray, real number, set.

Most important Defined Terms: area, parallel lines, parallelogram, right angle, triangle, congruent triangles, similar triangles, tangent to a circle, area.

Other Defined terms: acute angle, alternate angles, angle bisector, arc, area of a disc, base and corresponding apex and height of triangle or parallelogram, chord, circle, circumcentre, circumcircle, circumference of a circle, circumradius, collinear points, concurrent lines, convex quadrilateral, corresponding angles, diameter, disc, distance, equilateral triangle, exterior angles of a triangle, full angle, hypotenuse, incentre, incircle, inradius, interior opposite angles, isosceles triangle, median lines, midpoint of a segment, null angle, obtuse angle, perpendicular bisector of a segment, perpendicular lines, point of contact of a tangent, polygon, quadrilateral, radius, ratio, rectangle, reflex angle ordinary angle, rhombus, right-angled triangle, scalene triangle,

sector, segment, square, straight angle, subset, supplementary angles, transversal line, vertically-opposite angles.

Definable terms used without explicit definition: angles, adjacent sides, arms or sides of an angle, centre of a circle, endpoints of segment, equal angles, equal segments, line passes through point, opposite sides or angles of a quadrilateral, or vertices of triangles or quadrilaterals, point lies on line, side of a line, side of a polygon, the side opposite an angle of a triangle, vertex, vertices (of angle, triangle, polygon).

6   The Theory

Line[3] is short for straight line. Take a fixed **plane**[4], once and for all, and consider just lines that lie in it. The plane and the lines are **sets**[5] of **points**[6]. Each line is a **subset** of the plane, i.e. each element of a line is a point of the plane. Each line is endless, extending forever in both directions. Each line has infinitely-many points. The points on a line can be taken to be ordered along the line in a natural way. As a consequence, given any three distinct points on a line, exactly one of them lies **between** the other two. Points that are not on a given line can be said to be on one or other **side** of the line. The sides of a line are sometimes referred to as **half-planes**.

Notation 1. We denote points by roman capital letters A, B, C, etc., and lines by lower-case roman letters l, m, n, etc.

Axioms are statements we will accept as true[7].

Axiom 1 (Two Points Axiom). *There is exactly one line through any two given points. (We denote the line through A and B by AB.)*

Definition 1. The line **segment** $[AB]$ is the part of the line AB between A and B (including the endpoints). The point A divides the line AB into two pieces, called **rays**. The point A lies between all points of one ray and all

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[3]Line is undefined.

[4]Undefined term

[5]Undefined term

[6]Undefined term

[7] An **axiom** is a statement accepted without proof, as a basis for argument. A **theorem** is a statement deduced from the axioms by logical argument.

points of the other. We denote the ray that starts at A and passes through B by $[AB$. Rays are sometimes referred to as **half-lines**.

Three points usually determine three different lines.

Definition 2. If three or more points lie on a single line, we say they are **collinear**.

Definition 3. Let A, B and C be points that are not collinear. The **triangle** $\triangle ABC$ is the piece of the plane enclosed by the three line segments $[AB]$, $[BC]$ and $[CA]$. The segments are called its **sides**, and the points are called its **vertices** (singular **vertex**).

6.1   Length and Distance

We denote the set of all **real numbers**[8] by $\mathbb{R}$.

Definition 4. We denote the **distance**[9] between the points A and B by $|AB|$. We define the **length** of the segment $[AB]$ to be $|AB|$.

We often denote the lengths of the three sides of a triangle by a, b, and c. The usual thing for a triangle $\triangle ABC$ is to take $a = |BC|$, i.e. the length of the side opposite the vertex A, and similarly $b = |CA|$ and $c = |AB|$.

Axiom 2 (Ruler Axiom[10])**.** *The distance between points has the following properties:*

1. *the distance $|AB|$ is never negative;*

2. *$|AB| = |BA|$;*

3. *if C lies on AB, between A and B, then $|AB| = |AC| + |CB|$;*

4. *(marking off a distance) given any ray from A, and given any real number $k \geq 0$, there is a unique point B on the ray whose distance from A is k.*

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[8]Undefined term

[9]Undefined term

[10] Teachers used to traditional treatments that follow Euclid closely should note that this axiom (and the later Protractor Axiom) guarantees the existence of various points (and lines) without appeal to postulates about constructions using straight-edge and compass. They are powerful axioms.

Definition 5. The **midpoint** of the segment $[AB]$ is the point M of the segment with [11]

$$|AM| = |MB| = \frac{|AB|}{2}.$$

6.2   Angles

Definition 6. A subset of the plane is **convex** if it contains the whole segment that connects any two of its points.

For example, one side of any line is a convex set, and triangles are convex sets.

We do not define the term angle formally. Instead we say: There are things called **angles**. To each angle is associated:

1. a unique point A, called its **vertex**;

2. two rays $[AB$ and $[AC$, both starting at the vertex, and called the **arms** of the angle;

3. a piece of the plane called the **inside** of the angle.

An angle is either a null angle, an ordinary angle, a straight angle, a reflex angle or a full angle, Unless otherwise specified, you may take it that any angle we talk about is an ordinary angle.

Definition 7. An angle is a **null angle** if its arms coincide with one another and its inside is the empty set.

Definition 8. An angle is an **ordinary angle** if its arms are not on one line, and its inside is a convex set.

Definition 9. An angle is a **straight angle** if its arms are the two halves of one line, and its inside is one of the sides of that line.

Definition 10. An angle is a **reflex angle** if its arms are not on one line, and its inside is not a convex set.

Definition 11. An angle is a **full angle** if its arms coincide with one another and its inside is the rest of the plane.

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[11] Students may notice that the first equality implies the second.

Definition 12. Suppose that A, B, and C are three noncollinear points. We denote the (ordinary) angle with arms $[AB$ and $[AC$ by $\angle BAC$ (and also by $\angle CAB$). We shall also use the notation $\angle BAC$ to refer to straight angles, where A, B, C are collinear, and A lies between B and C (either side could be the inside of this angle).

Sometimes we want to refer to an angle without naming points, and in that case we use lower-case greek letters, α,β,γ, etc.

6.3   Degrees

Notation 2. We denote the number of **degrees** in an angle $\angle BAC$ or α by the symbol $|\angle BAC|$, or $|\angle \alpha|$, as the case may be.

Axiom 3 (Protractor Axiom)**.** *The number of degrees in an angle (also known as its degree-measure) is always a number between* $0°$ *and* $360°$*. The number of degrees of an ordinary angle is less than* $180°$*. It has these properties:*

1. *A straight angle has* $180°$*.*

2. *Given a ray* $[AB$*, and a number d between* 0 *and* 180*, there is exactly one ray from A on each side of the line AB that makes an (ordinary) angle having d degrees with the ray* $[AB$*.*

3. *If D is a point inside an angle $\angle BAC$, then*

$$|\angle BAC| = |\angle BAD| + |\angle DAC|.$$

Null angles are assigned $0°$, full angles $360°$, and reflex angles have more than $180°$. To be more exact, if A, B, and C are noncollinear points, then the reflex angle "outside" the angle $\angle BAC$ measures $360° - |\angle BAC|$, in degrees.

Definition 13. The ray $[AD$ is the **bisector** of the angle $\angle BAC$ if

$$|\angle BAD| = |\angle DAC| = \frac{|\angle BAC|}{2}.$$

We say that an angle is 'an angle of' (for instance) $45°$, if it has 45 degrees in it.

Definition 14. A **right angle** is an angle of exactly $90°$.

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Definition 15. An angle is **acute** if it has less than 90°, and **obtuse** if it has more than 90°.

Definition 16. If $\angle BAC$ is a straight angle, and D is off the line BC, then $\angle BAD$ and $\angle DAC$ are called **supplementary angles**. They add to 180°.

Definition 17. When two lines AB and AC cross at a point A, they are **perpendicular** if $\angle BAC$ is a right angle.

Definition 18. Let A lie between B and C on the line BC, and also between D and E on the line DE. Then $\angle BAD$ and $\angle CAE$ are called **vertically-opposite angles**.
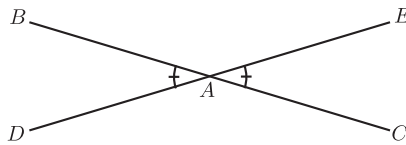
Figure 1.

Theorem 1 (Vertically-opposite Angles).
Vertically opposite angles are equal in measure.

Proof. See Figure 1. The idea is to add the same supplementary angles to both, getting 180°. In detail,

$$|\angle BAD| + |\angle BAE| \;=\; 180°,$$
$$|\angle CAE| + |\angle BAE| \;=\; 180°,$$

so subtracting gives:

$$|\angle BAD| - |\angle CAE| \;=\; 0°,$$
$$|\angle BAD| \;=\; |\angle CAE|.$$

$\square$

6.4 Congruent Triangles

Definition 19. Let A, B, C and A', B', C' be triples of non-collinear points. We say that the triangles ΔABC and $\Delta A'B'C'$ are **congruent** if all the sides and angles of one are equal to the corresponding sides and angles of the other, i.e. $|AB| = |A'B'|$, $|BC| = |B'C'|$, $|CA| = |C'A'|$, $|\angle ABC| = |\angle A'B'C'|$, $|\angle BCA| = |\angle B'C'A'|$, and $|\angle CAB| = |\angle C'A'B'|$. See Figure 2.
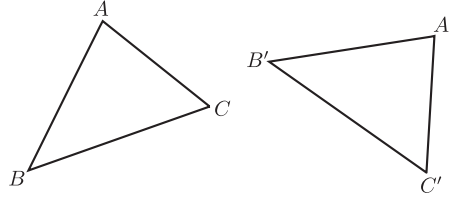
Figure 2.

Notation 3. Usually, we abbreviate the names of the angles in a triangle, by labelling them by the names of the vertices. For instance, we write $\angle A$ for $\angle CAB$.

Axiom 4 (SAS+ASA+SSS[12]).
If (1) $|AB| = |A'B'|$, $|AC| = |A'C'|$ and $|\angle A| = |\angle A'|$,
or
(2) $|BC| = |B'C'|$, $|\angle B| = |\angle B'|$, and $|\angle C| = |\angle C'|$,
or
(3) $|AB| = |A'B'|$, $|BC| = |B'C'|$, and $|CA| = |C'A'|$
then the triangles $\triangle ABC$ and $\triangle A'B'C'$ are congruent.

Definition 20. A triangle is called **right-angled** if one of its angles is a right angle. The other two angles then add to 90°, by Theorem 4, so are both acute angles. The side opposite the right angle is called the **hypotenuse**.

Definition 21. A triangle is called **isosceles** if two sides are equal[13]. It is **equilateral** if all three sides are equal. It is **scalene** if no two sides are equal.

Theorem 2 (Isosceles Triangles)**.**
(1) In an isosceles triangle the angles opposite the equal sides are equal.
(2) Conversely, If two angles are equal, then the triangle is isosceles.

Proof. (1) Suppose the triangle $\triangle ABC$ has $AB = AC$ (as in Figure 3). Then $\triangle ABC$ is congruent to $\triangle ACB$ [SAS]
$\therefore \angle B = \angle C$.

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[12]It would be possible to prove all the theorems using a weaker axiom (just SAS). We use this stronger version to shorten the course.
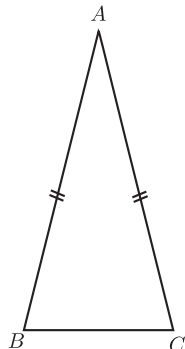
[13] The simple "equal" is preferred to "of equal length"

Figure 3.

(2) Suppose now that $\angle B = \angle C$. Then
ΔABC is congruent to ΔACB [ASA]
$\therefore |AB| = |AC|, \Delta ABC$ is isosceles.   □

Acceptable Alternative Proof of (1). Let D be the midpoint of $[BC]$, and use SAS to show that the triangles ΔABD and ΔACD are congruent. (This proof is more complicated, but has the advantage that it yields the extra information that the angles $\angle ADB$ and $\angle ADC$ are equal, and hence both are right angles (since they add to a straight angle)).   □

6.5 Parallels

Definition 22. Two lines l and m are **parallel** if they are either identical, or have no common point.

Notation 4. We write $l||m$ for "l is parallel to m".

Axiom 5 (Axiom of Parallels). *Given any line l and a point P, there is exactly one line through P that is parallel to l.*

Definition 23. If l and m are lines, then a line n is called a **transversal** of l and m if it meets them both.

Definition 24. Given two lines AB and CD and a transversal BC of them, as in Figure 4, the angles $\angle ABC$ and $\angle BCD$ are called **alternate** angles.
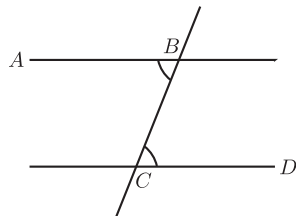
Figure 4.

Theorem 3 (Alternate Angles)**.** *Suppose that A and D are on opposite sides of the line BC.*
(1) If $|\angle ABC| = |\angle BCD|$, then $AB||CD$. In other words, if a transversal makes equal alternate angles on two lines, then the lines are parallel.
(2) Conversely, if $AB||CD$, then $|\angle ABC| = |\angle BCD|$. In other words, if two lines are parallel, then any transversal will make equal alternate angles with them.
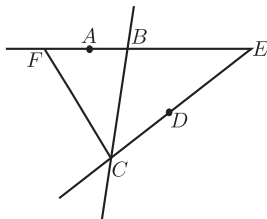

Figure 5.

Proof. (1) Suppose $|\angle ABC| = |\angle BCD|$. If the lines AB and CD do not meet, then they are parallel, by definition, and we are done. Otherwise, they meet at some point, say E. Let us assume that E is on the same side of BC as D.[14] Take F on EB, on the same side of BC as A, with $|BF| = |CE|$ (see Figure 5). [Ruler Axiom]

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[14]Fuller detail: There are three cases:
$1°$: E lies on BC. Then (using Axiom 1) we must have $E = B = C$, and $AB = CD$.
$2°$: E lies on the same side of BC as D. In that case, take F on EB, on the same side of BC as A, with $|BF| = |CE|$. [Ruler Axiom]
Then $\triangle BCE$ is congruent to $\triangle CBF$. [SAS]
Thus
$$|\angle BCF| = |\angle CBE| = 180° - |\angle ABC| = 180° - |\angle BCD|,$$

Then ΔBCE is congruent to ΔCBF.   [SAS]
Thus

$$|\angle BCF| = |\angle CBE| = 180° - |\angle ABC| = 180° - |\angle BCD|,$$

so that F lies on DC.   [Ruler Axiom]
Thus AB and CD both pass through E and F, and hence coincide,
                                                          [Axiom 1]
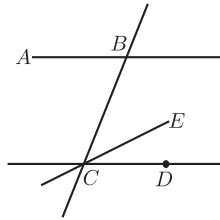Hence AB and CD are parallel.   [Definition of parallel]

Figure 6.

(2) To prove the converse, suppose $AB||CD$. Pick a point E on the same side of BC as D with $|\angle BCE| = |\angle ABC|$. (See Figure 6.) By Part (1), the line CE is parallel to AB. By Axiom 5, there is only one line through C parallel to AB, so $CE = CD$. Thus $|\angle BCD| = |\angle BCE| = |\angle ABC|$.   □

Theorem 4 (Angle Sum 180). *The angles in any triangle add to* $180°$.
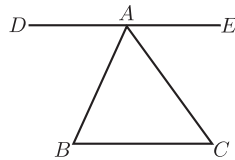

Figure 7.

so that F lies on DC.   [Protractor Axiom]
Thus AB and CD both pass through E and F, and hence coincide.   [Axiom 1]
3°: E lies on the same side of BC as A. Similar to the previous case.
Thus, in all three cases, $AB = CD$, so the lines are parallel.

Proof. Let $\triangle ABC$ be given. Take a segment $[DE]$ passing through A, parallel to BC, with D on the opposite side of AB from C, and E on the opposite side of AC from B (as in Figure 7). $\hspace{2cm}$ [Axiom of Parallels]
Then AB is a transversal of DE and BC, so by the Alternate Angles Theorem,

$$|\angle ABC| = |\angle DAB|.$$

Similarly, AC is a transversal of DE and BC, so

$$|\angle ACB| = |\angle CAE|.$$

Thus, using the Protractor Axiom to add the angles,

$$\begin{aligned} &|\angle ABC| + |\angle ACB| + |\angle BAC| \\ = \ &|\angle DAB| + |\angle CAE| + |\angle BAC| \\ = \ &|\angle DAE| = 180°, \end{aligned}$$

since $\angle DAE$ is a straight angle. $\hspace{4cm}$ $\square$

Definition 25. Given two lines AB and CD, and a transversal AE of them, as in Figure 8(a), the angles $\angle EAB$ and $\angle ACD$ are called **corresponding** angles[15].
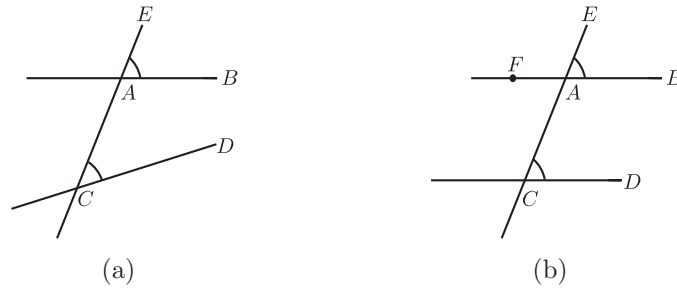

(a) $\hspace{6cm}$ (b)

Figure 8.

Theorem 5 (Corresponding Angles)**.** *Two lines are parallel if and only if for any transversal, corresponding angles are equal.*

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[15]with respect to the two lines and the given transversal.

Proof. See Figure 8(b). We first assume that the corresponding angles $\angle EAB$ and $\angle ACD$ are equal. Let F be a point on AB such that F and B are on opposite sides of AE. Then we have

$|\angle EAB| = |\angle FAC|$ [Vertically opposite angles]

Hence the alternate angles $\angle FAC$ and $\angle ACD$ are equal and therefore the lines $FA = AB$ and CD are parallel.

For the converse, let us assume that the lines AB and CD are parallel. Then the alternate angles $\angle FAC$ and $\angle ACD$ are equal. Since

$|\angle EAB| = |\angle FAC|$ [Vertically opposite angles]

we have that the corresponding angles $\angle EAB$ and $\angle ACD$ are equal.   □

Definition 26. In Figure 9, the angle α is called an **exterior angle** of the triangle, and the angles β and γ are called (corresponding) **interior opposite angles**.[16]
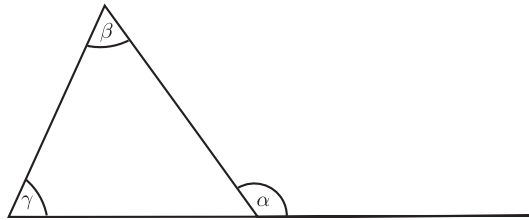

Figure 9.

Theorem 6 (Exterior Angle). *Each exterior angle of a triangle is equal to the sum of the interior opposite angles.*

Proof. See Figure 10. In the triangle ΔABC let α be an exterior angle at A. Then

$|\alpha| + |\angle A| = 180°$ [Supplementary angles]

and

$|\angle B| + |\angle C| + |\angle A| = 180°.$ [Angle sum 180°]

Subtracting the two equations yields $|\alpha| = |\angle B| + |\angle C|$.   □

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[16]The phrase **interior remote angles** is sometimes used instead of **interior opposite angles**.