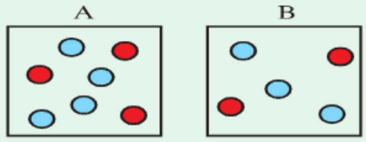
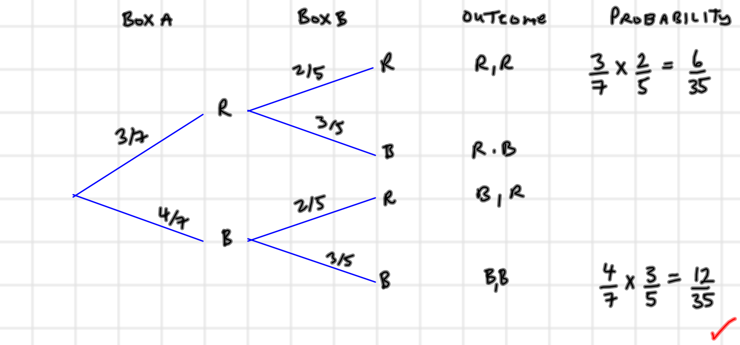
Example 1

Box A contains 3 red beads and 4 blue beads
Box B contains 2 red beads and 3 blue beads
One bead is taken at random from each box.
(i) Draw a tree diagram to show all the outcomes.
(ii) Work out the probability that they both will have the same colour.



Tree diagram

(i)

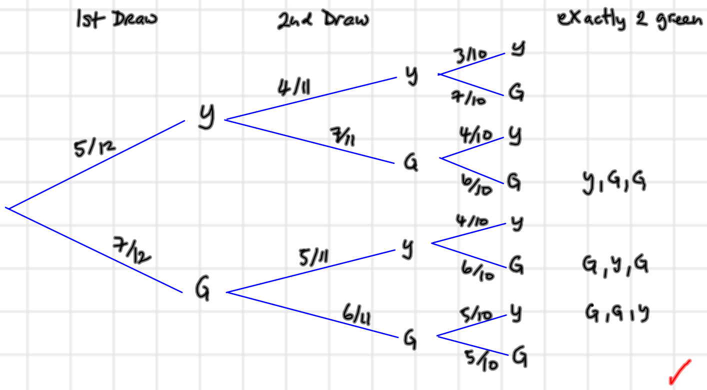


(ii) $P(R,R \text{ or } B,B) = \frac{6}{35} + \frac{12}{35} = \frac{18}{35}$ ✓

Example 2

A box contains 12 beads. Five are yellow and the rest are green. A bead is removed from the box and its colour is noted. It is not returned to the box. A second selection is then made and the process is repeated, followed by a third selection.
(i) Draw a tree diagram outlining this situation
(ii) Find the probability of selecting exactly two green beads.

5 yellow
7 green
at start



$P(\text{exactly 2 green}) = P(y,g,g \text{ or } g,y,g \text{ or } g,g,y)$

$= \left(\frac{5}{12} \times \frac{7}{11} \times \frac{6}{10}\right) + \left(\frac{7}{12} \times \frac{5}{11} \times \frac{6}{10}\right) + \left(\frac{7}{12} \times \frac{6}{11} \times \frac{5}{10}\right)$

$= \frac{7}{44} + \frac{7}{44} + \frac{7}{44} = \frac{21}{44}$ ✓

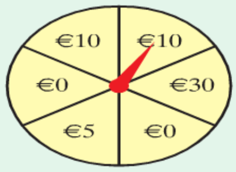
Example 1

This circle is divided into 6 equal sectors.

You pay €10 to spin the arrow and you win the amount in the sector where the arrow stops.

What is the expected amount you win or lose in this game?

We find the expected value of the payout by constructing the table below which shows the probability of each outcome.



Payout (x)	Probability (P)	Payout × Probability ($x \times P$)
€0	$2/6 = 1/3$	$0 = €0$
€5	$1/6$	$5/6 = €5/6$
€10	$2/6 = 1/3$	$10/3 = €10/3$
€30	$1/6$	$30/6 = €5$

$$\text{Sum} = €\left(0 + 5/6 + 10/3 + 5\right) = 9.1\dot{6} \approx €9.17$$

$$\text{expect} \quad €(9.17 - 10) = -€0.83$$

Example 2

Luke rolls a pair of dice in a game of chance that costs €2 to play.

The table on the right gives the financial outcome for each event.

(i) Calculate the financial expectation for this game.

(ii) Is this game fair? Explain your answer.

Event	Financial outcome
Any double	Win €4
Total of 7	Win €3
Odd sum (except 7)	Money back (€2)
Even sum (except doubles)	Lose €3

SAMPLE SPACE

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

PROBABILITY	€ PROBABILITY x PAYOUT
$P(\text{double}) = 6/36 = 1/6$	$\frac{1}{6} \times 4 = 2/3$
$P(7) = 6/36 = 1/6$	$\frac{1}{6} \times 3 = 1/2$
$P(\text{odd sum except } 7) = 12/36 = 1/3$	$\frac{1}{3} \times 2 = 2/3$
$P(\text{even sum except doubles}) = 12/36 = 1/3$	$\frac{1}{3} \times (-3) = -1$
Sum	$€\,5/6 = 0.83$

expect win/lose ; $€(0.83 - 2) = -€1.17$

It's not fair because financial expectation is not 0.

Example 1

An unbiased die is thrown 5 times. Find the probability of obtaining
(i) 1 six (ii) 3 sixes (iii) at least 1 six.

$p(r \text{ successes}) = \binom{n}{r} p^r q^{n-r}$

p = probability success
q = probability failure

Success = throw 6 $p = 1/6$
fail = don't throw 6 $q = 5/6$

(i) $p(1 \text{ Success in 6 throws}) = \binom{5}{1} p^1 q^4$

$n=5,\ r=1$

$= \binom{5}{1}\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^4 = \frac{3125}{7776}$

(ii) $p(3 \text{ successes}) = \binom{5}{3} p^3 q^2$

$n=5,\ r=3$

$= \binom{5}{3}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^2 = \frac{125}{3888}$

$P(\text{at least 1 success}) = 1 - p(\text{no Success})$

$= 1 - \binom{5}{0} p^0 q^5 = 1 - \left(\frac{5}{6}\right)^5 = \frac{4651}{7776}$

Example 2

Given that 10% of apples are bad, find the probability that in a box containing 6 apples, there is
(i) no bad apple
(ii) just one bad apple
(iii) at least one bad apple.

$p(r \text{ successes}) = \binom{n}{r} p^r q^{n-r}$

p = probability success
q = probability failure

Success = bad apple , $p = 1/10$
fail = not bad , $q = 9/10$

6 apples in a box $\Rightarrow n=6$

(i) $p(\text{no Success}) = \binom{6}{0} p^0 q^6 = q^6 = \left(\frac{9}{10}\right)^6$

(ii) $p(1 \text{ Success}) = \binom{6}{1} p^1 q^5 = 6\left(\frac{1}{10}\right)\left(\frac{9}{10}\right)^5 = \frac{6(9)^5}{10^6}$

(iii) $P(\text{at least 1 bad}) = 1 - P(\text{no bad}) = 1 - \left(\frac{9}{10}\right)^6$

Example 3

A card is drawn at random from a normal deck of playing cards and then replaced. The process is repeated until the third diamond appears.
Find the probability that this happens when the tenth card is drawn.

$p = P(\text{success}) = \frac{1}{4}$

$q = P(\text{fail}) = \frac{3}{4}$

$P(\text{2 diamonds in 9 trials } \underline{\text{and}} \text{ diamond in 10th trial})?$

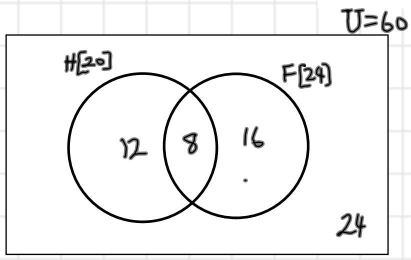
$P(r=2, n=9) = \binom{9}{2} p^2 q^7 = \binom{9}{2}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^7 = 0.3$

$P(\text{diamond on 10th trial}) = 0.25$

$P(\text{Both}) = (0.3)(0.25) = 0.075$

Example 1

In a group of 60 students, 20 study History, 24 study French, 8 study both History and French and 24 study neither.
Illustrate this information on a Venn diagram.
Now investigate if the events 'a student studies History' and 'a student studies French' are independent.



Independent?

If independent

$P(A) \times P(B) = P(A \cap B)$

Is $P(H \cap F) = P(H) \times P(F)$

$P(H \cap F) = 8/60 = 2/15$

$P(H) \times P(F) = (20/60) \times (24/60) = \frac{1}{3} \times \frac{2}{5} = \frac{2}{15}$

$\Rightarrow$ Independent

Example 2

Two events A and B are such that $P(A) = 0.5$, $P(B) = 0.4$ and $P(A|B) = 0.3$.
(i) Find $P(A \cap B)$.
(ii) Investigate whether or not the events A and B are independent.

(i) $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$

$\Rightarrow P(A|B) \times P(B) = P(A \cap B)$
$= (0.3)(0.4) = 0.12$

(ii) Independent?

$P(A) \times P(B) = (0.4)(0.5) = 0.2$

$\Rightarrow P(A) \times P(B) \neq P(A \cap B)$
$0.2 \neq 0.12$

$\Rightarrow$ not independent

Example 3

Two ordinary fair dice, one red and one blue, are to be rolled once.
(i) Find the probability of the following events:
Event A: the number showing on the red dice will be a 5 or a 6.
Event B: the total of the numbers showing on the two dice will be 7.
Event C: the total of the numbers showing on the two dice will be 8.
(ii) Show that events A and B are independent.
(iii) Investigate if events A and C are independent.

Red

Blue	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

(i) $P(A) = \frac{2}{6} = \frac{1}{3}$
$P(B) = \frac{6}{36} = \frac{1}{6}$
$P(C) = \frac{5}{36}$ ✓

Check Independence

$P(A) \times P(B) = P(A \cap B)$
if independent

(ii) $P(A \cap B) = P(\text{Red 5 or 6} \cap \text{total } 7) = \frac{2}{36} = \frac{1}{18}$
$P(A) \times P(B) = \left(\frac{1}{3}\right)\left(\frac{1}{6}\right) = \frac{1}{18}$
$\Rightarrow P(A) \times P(B) = P(A \cap B)$
$\Rightarrow$ events are independent ✓

(iii) $P(A \cap C) = P(\text{Red 5 or 6} \cap \text{total } 8) = \frac{2}{36} = \frac{1}{18}$
$P(A) \times P(C) = \left(\frac{1}{3}\right)\left(\frac{5}{36}\right) = \frac{5}{108}$
$\Rightarrow P(A) \times P(C) \neq P(A \cap C)$
$\Rightarrow$ events are not independent ✓

Example 1

In Class 6A, two boys and four girls study music.
In Class 6B, four boys and six girls study music.
Two pupils are chosen at random from each of the two classes to perform at a concert.
(i) In how many ways can the 4 pupils be selected?
(ii) Calculate the probability that the four chosen consist of 2 boys from 6A and 2 girls from 6B.
(iii) Calculate the probability that the four pupils are of the same gender.

(i)

CLASS A 6	
2	4
Boy	Girl

CLASS B 10	
4	6
Boy	Girl

(i) ways of choosing 2 from each class $= \binom{6}{2} \times \binom{10}{2} = 675$

(ii) P (2 Boys 6A and 2 girls 6B) ?

ways of choosing $= \binom{2}{2} \times \binom{6}{2} = 15$

$P(2 \text{ Boys } 6A \cap 2 \text{ girls } 6B) = \frac{15}{675}$

(iii) $P(\text{all boys (or) all girls}) = \dfrac{\binom{2}{2}\binom{4}{2} + \binom{4}{2}\binom{6}{2}}{675} = \frac{32}{225}$

Example 2

Three cards are drawn at random, and without replacement, from a pack of 52 playing cards. Find the probability that
(i) the three cards drawn are the Jack of spades, the Queen of clubs and the King of clubs
(ii) the three cards are aces
(iii) two cards are red and the third one is a club
(iv) the three cards are of the same colour.

Ways of selecting 3 cards ? $= \binom{52}{3} = 22{,}100$

(i) Ways of selecting 3 particular cards is $\binom{3}{3} = 1$

P(Jack of spades, Queen of clubs, King clubs (any order)) $= \frac{1}{22{,}100}$

(ii) Ways of selecting 3 aces $= \binom{4}{3} = 4$

$P(3 \text{ aces}) = \frac{4}{22{,}100} = \frac{1}{5525}$

(iii) Ways of selecting 2 reds and 1 club $= \binom{26}{2} \times \binom{13}{1} = 4225$

$P(2 \text{ Red and } 1 \text{ club}) = \frac{4225}{22100}$

(iv) $P(\text{all Red or all black}) = \dfrac{\binom{26}{3} + \binom{26}{3}}{22100} = \frac{4}{17}$

Example 1

Milk cartons are designed to hold 600 ml of milk.
Their capacities are normally distributed with mean 600 ml and standard deviation 10 ml.

(i) Calculate the standard scores for these capacities
(a) 610 ml (b) 595 ml (c) 625 ml.

(ii) Mark these z-scores on the diagram of a standard normal curve.

(iii) What can you conclude about a carton containing 625 ml?

$\mu = 600$ ml $\quad \sigma = 10$ ml

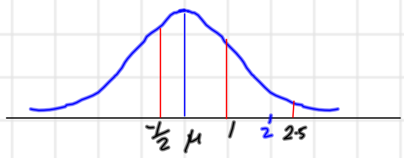
(i) z-scores?

(a) 610 ml $= \mu + \sigma \Rightarrow z = 1$

(b) 595 ml $= \mu - \frac{\sigma}{2} \Rightarrow z = -\frac{1}{2}$

(c) 625 ml $= \mu + 2.5\sigma \Rightarrow z = 2\frac{1}{2}$

(ii) Show on normal curve



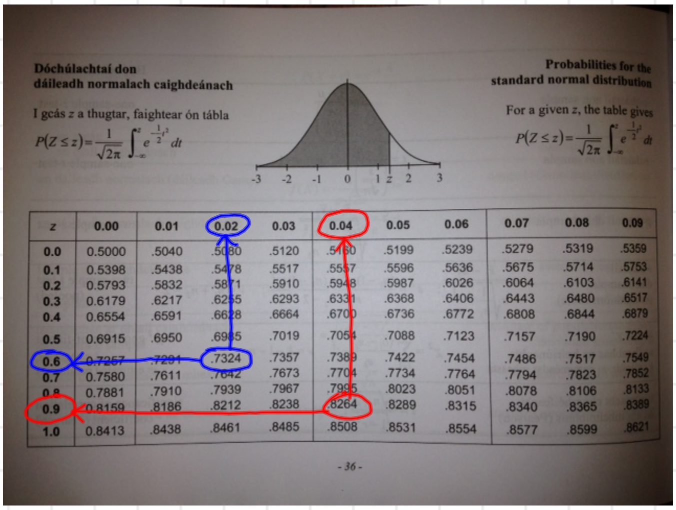
(iii) 625 ml is outside 95% limit of 620 ml

Example 2

Find a in each of the following:
(i) $P(z \leqslant a) = 0.7324$ (ii) $P(z \leqslant a) = 0.1724$

Read directly from table

(i) $a = 0.62$



Dóchúlachtaí don dáileadh normalach caighdeánach

I gcás z a thugtar, faightear ón tábla

$P(Z \le z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}t^2}\, dt$

Probabilities for the standard normal distribution

For a given z, the table gives

$P(Z \le z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}t^2}\, dt$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	0.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	0.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	0.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	0.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	0.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	0.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	0.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	0.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	0.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	0.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621

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$P(z \le a) = 0.1724 \Rightarrow a$ is negative

$P(z \le -a) = 1 - 0.1724 = 0.8276$

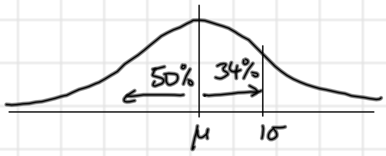
$\Rightarrow -a = 0.94 \quad \Rightarrow a = -0.94$

Example 3

The mean height of all the students in a certain school is 175 cm and the standard deviation is 15 cm. If a student is selected at random, find the probability that he is less than, or equal to, 190 cm tall.

$\mu = 175$ cm , $\sigma = 15$ cm   190 cm $= \mu + \sigma$

empirical Rule
68% ($\mu \pm 1\sigma$)

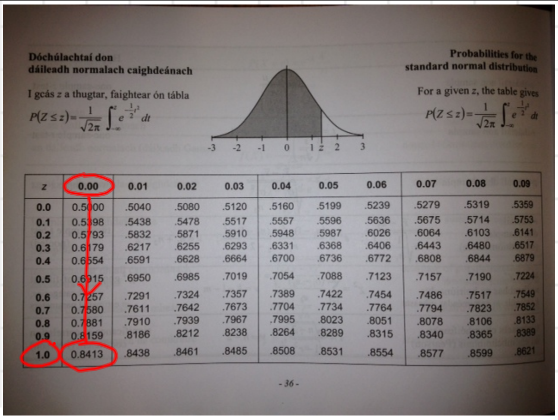


$\Rightarrow P(\text{less than } 190\text{cm}) = 84\%$

or use tables

$z = 1$

$P(\leq 190\text{cm}) = 0.8413$



Dóchúlachtaí don dáileadh normalach caighdeánach

I gcás z a thugtar, faightear ón tábla

$$P(Z \leq z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}t^2}\, dt$$

Probabilities for the standard normal distribution

For a given z, the table gives

$$P(Z \leq z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}t^2}\, dt$$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	0.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	0.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	0.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	0.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	0.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	0.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	0.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	0.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	0.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	0.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621

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Example 1

With every breakfast cereal pack there is a free coloured toy. There are 6 different colours of toys.
How many packs of breakfast cereal do you need to buy to collect the 6 different-coloured toys?

Describe a simulation you could design to give an approximation of the answer?

Let each toy be represented by the number on a dice.

Roll until all numbers appear.

Repeat simulation a number of times.

Example 2

In families that have 5 children, we wish to investigate the probability that boys outnumber girls.
Devise a simulation of this situation and use it to determine whether the probability is more than, less than, or equal to $\frac{1}{2}$.

Assume equal chance of boy or girl being born.

Repeatedly toss 5 coins,

Let H stand for boy

Let T stand for girl

Table the Results and draw conclusion

Example 3

Four people are selected at random from a large crowd.
What is the probability that two or more of them have birthdays in the same month?

(i) Devise a simulation that could determine the probability of this happening.

(ii) Run the simulation 100 times and approximate the probability.

Use a random generator with numbers 1 to 12 to represent the months of the year.

Repeatedly generate 4 numbers

Table the results and draw conclusion.