Integration

Integration is the reverse of differentiation.

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$$

Increase the power by one and divide by the new power

c is the constant of integration

$$\int 3x^2 = \frac{3x^3}{3} + c = x^3 + c$$

$$\int \frac{1}{x^3} = \int x^{-3} + c = \frac{x^{-2}}{-2} + c = -\frac{1}{2x^2} + c$$

$$\int \frac{1}{\sqrt{x}} = \int x^{-\frac{1}{2}} = \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c = 2x^{\frac{1}{2}} + c = 2\sqrt{x} + c$$

Basic Rules

$$\int \sin(nx+k)\,dx = -\frac{\cos\,(nx+k)}{n} + c$$

$$\int \cos(nx+k)\,dx = \frac{\sin(nx+k)}{n} + c$$

$$\int \frac{1}{ax+b}\,dx = \frac{\ln(ax+b)}{a} + c$$

$$\int e^{ax+b}dx = \frac{e^{ax+b}}{a} + c$$

$$\int \sin(4x+6)\,dx = -\frac{\cos\,(4x+6)}{4} + c$$

$$\int \frac{1}{3x-2}\,dx = \frac{\ln(3x-2)}{3} + c$$

$$\int e^{-3x+2}dx = \frac{e^{-3x+2}}{-3} + c$$

Substitution

There is no product or quotient rule in Integration so we use substitution.

$$\int_0^{\sqrt{5}} \frac{x}{\sqrt{x^2+4}}\,dx$$

Substitution

$u = x^2 + 4$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$$\frac{1}{2}\int \frac{1}{\sqrt{u}}\,du$$

$$= \frac{1}{2}\int u^{-\frac{1}{2}}\,du$$

$$= \frac{1}{2}\frac{\mathrm{u}^{\frac{1}{2}}}{\frac{1}{2}}$$

$$= \frac{1}{2}2\mathrm{u}^{\frac{1}{2}}$$

$$= \mathrm{u}^{\frac{1}{2}}$$

$$= \sqrt{\mathrm{u}}$$

$$= \sqrt{x^2+4}$$

$$\int_0^{\sqrt{5}} \frac{x}{\sqrt{x^2+4}}\,dx = \left(\sqrt{x^2+4}\right)_0^{\sqrt{5}}$$

$$= \sqrt{\left(\sqrt{5}\right)^2 + 4} - \sqrt{(0)^2+4}$$

$$= \sqrt{\left(\sqrt{5}\right)^2 + 4} - \sqrt{(0)^2+4}$$

$$= 3 - 2 = 1$$

Inverse Trigonometric Integrals

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + c$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$$

$$\int \frac{dx}{\sqrt{25-x^2}}$$

$$= \sin^{-1}\frac{x}{5} + c$$

$$\int \frac{dx}{16+9x^2}$$

$$\int \frac{dx}{4^2+(3x)^2}$$

$$= \frac{\frac{1}{4}\tan^{-1}\frac{3x}{4}}{3}$$

$$= \frac{1}{12}\tan^{-1}\frac{3x}{4}$$

$$\int \sqrt{\boldsymbol{a^2 - x^2}}\,\boldsymbol{dx}$$

Special Case

With these let $x = asin\theta$ and change to a new integral in θ. You must also change the limits. Learn example in proofs handout.

Completing the Square

$$\int \frac{dx}{\sqrt{5-4x-x^2}}$$

$$= \int \frac{dx}{\sqrt{4+5-(4+4x+x^2)}}$$

$$= \int \frac{dx}{\sqrt{9-(x+2)^2}}$$

$$\int \frac{dx}{\sqrt{3^2-(x+2)^2}}$$

$$= \sin^{-1}\frac{x+2}{3} + c$$

Powers of cos and sin

$$\cos^2 A = \frac{1}{2}(1 + \cos 2A)$$
$$\sin^2 A = \frac{1}{2}(1 - \cos 2A)$$

To integrate even powers of $cos\ x$ and $sin\ x$, rewrite the integrand using the double angle identities of the tables.

$$\int_0^{\frac{\pi}{4}} \sin^2 x\, dx$$
$$= \int_0^{\frac{\pi}{4}} \frac{1}{2}(1 - \cos 2x)\, dx$$
$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} (1 - \cos 2x)\, dx$$
$$= \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\frac{\pi}{4}}$$
$$= \frac{1}{2}\left[\left(\left(\frac{\pi}{4}\right) - \frac{\sin 2\left(\frac{\pi}{4}\right)}{2}\right) - \left((0) - \frac{\sin 2(0)}{2}\right)\right]$$
$$= \frac{1}{2}\left[\left(\frac{\pi}{4} - \frac{1}{2}\right) - \left(0 - \frac{0}{2}\right)\right]$$
$$= \frac{\pi}{8} - \frac{1}{4}$$

$$\int \cos^2 3x\, dx$$
$$= \int \frac{1}{2}(1 + \cos 6x)\, dx$$
$$= \frac{1}{2}\left(x + \frac{sin\ 6x}{6}\right) + c$$
$$= \frac{x}{2} + \frac{sin\ 6x}{12} + c$$

Changing Trigonometric Products to Sums

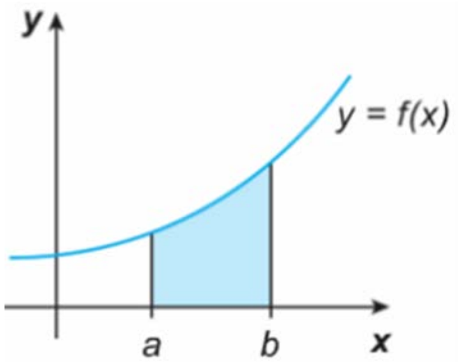
Always put the bigger angle 1st then use formula in tables to change product into a sum or difference.

$$\int \sin 6\theta \cos 4\theta\, d\theta$$

Use the $2sinAcosB = \sin(A+B) + \sin(A-B)$ formula in tables

$$\int \frac{1}{2}(\sin 10\theta + \sin 2\theta)\, d\theta$$
$$= \frac{1}{2}\left[-\frac{\cos 10\theta}{10} - \frac{cos 2\theta}{2}\right] + c$$
$$= -\frac{\cos 10\theta}{20} - \frac{cos 2\theta}{4} + c$$

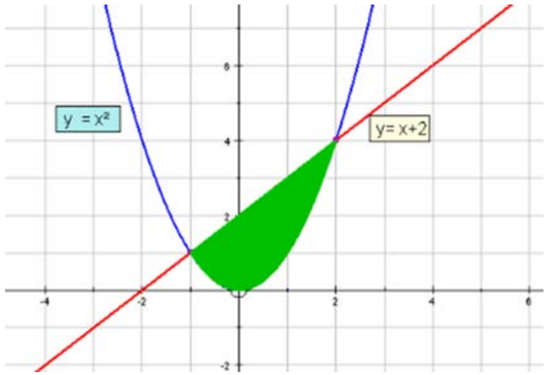
Area Bounded By Curve and Axis



The area enclosed by a curve $y = f(x)$ and the x axis between $x = a$ and $x = b$, is given by:

$$Area = \int_a^b f(x)\, dx$$

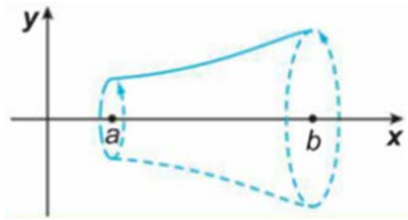
Area Between Curves



We need to find where the curves intersect using simultaneous equation.

Then subtract the areas under the curves between the points of intersection

Volume rotating about the $x - axis$ and $y - axis$



The volume generated by rotating a curve $y = f(x)$ about the axis from $x = a$ and $x = b$, is given by:

$$V_x = \pi \int_a^b y^2\, dx \qquad V_y = \pi \int_a^b x^2\, dy$$

Learn the special cases to generate formula for sphere and cone. (Proof Notes)