Leaving Cert Honours Maths

Integration

(Integral Calculus)

Differentiation = Division of changing terms

Integration = Multiplication of changing terms

Integration - Introduction

Integration is the opposite to differentiation

ADD / SUBTRACT

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

We do 3 things
- add 1 to power
- divide by new power
- add c

eg.1 $\int 3x^2 dx = \frac{3x^3}{3} + c = x^3$

eg.2 $\int \frac{1}{x^2} dx \quad \int x^{-2} = \frac{x^{-1}}{-1} + c = -\frac{1}{x} + c$

eg.3 $\int \frac{1}{\sqrt{x}} dx = \int x^{-\frac{1}{2}} dx = \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c = 2\sqrt{x} + c$

eg.4 $\int (2x + x^2) dx = \frac{2x^2}{2} + \frac{x^3}{3} + c = x^2 + \frac{x^3}{3} + c$

We do 3 things
- add 1 to power
- divide by new power
- add c

In these examples we have to 'prepare' them for integration

Prepare to integrate

Separate fraction

eg.5 $\int \frac{3+x}{\sqrt{x}}\,dx$

$\frac{x}{\sqrt{x}} = \frac{\sqrt{x}\sqrt{x}}{\sqrt{x}} = \sqrt{x}$

$= \int \left(\frac{3}{\sqrt{x}} + \frac{x}{\sqrt{x}}\right) dx = \int (3x^{-\frac{1}{2}} + x^{\frac{1}{2}})\,dx$

integrate

$\int x^n dx = \frac{x^{n+1}}{n+1} + C$

$= \frac{3x^{\frac{1}{2}}}{\frac{1}{2}} + \frac{x^{3/2}}{3/2} + c = 6x^{\frac{1}{2}} + \frac{2}{3}x^{3/2} + c$

eg.6 $\int \left(x + \frac{1}{x}\right)^2 dx$

Prepare to integrate

expand

$= \int \left(x^2 + 2(x)\left(\frac{1}{x}\right) + \left(\frac{1}{x}\right)^2\right) dx$

$= \int (x^2 + 2 + x^{-2})\,dx$

integrate

$\int x^n dx = \frac{x^{n+1}}{n+1} + C$

$= \frac{x^3}{3} + 2x + \frac{x^{-1}}{-1} + c$

Definite v Indefinite Integral

① Indefinite Integrals - no limits

$\int x^2\,dx$

integrate

add on +C

$= \frac{x^3}{3} + c$

② Definite Integrals - with limits

limits

$\int_1^2 x^2\,dx$

integrate

no need for +C

$= \left[\frac{x^3}{3}\right]_1^2$

evaluate

Sub in upper limit

minus sub in lower limit

$= \frac{(2)^3}{2} - \frac{(1)^3}{2} = 4 - \frac{1}{2} = 3\frac{1}{2}$

Multiply, Divide, chain Rule ⇒ Substitution Method

eg.1

$\int 2x(x^2+1)^4\,dx$

prepare substitution

let $u = x^2+1$

$\frac{du}{dx} = 2x$

$du = 2x\,dx$

Substitute

note: everything must be changed to u's

$= \int u^4\,du$

Integrate

$\int x^n dx = \frac{x^{n+1}}{n+1} + C$

$= \frac{u^5}{5} + c$

Present answer with x's

$= \frac{(x^2+1)^5}{5} + c$

Substitution method

eg.2

$\int \frac{x}{\sqrt{x-1}}\,dx = \int \frac{x}{(x-1)^{1/2}}\,dx$

prepare substitution

let $u = x-1$ also $x = u+1$

$\frac{du}{dx} = 1$

$du = dx$

Substitute

$= \int \frac{u+1}{u^{1/2}}\,du = \int\left(\frac{u}{u^{1/2}} + \frac{1}{u^{1/2}}\right)du = \int\left(u^{1/2} + u^{-1/2}\right)du$

Integrate

$\int x^n dx = \frac{x^{n+1}}{n+1} + C$

$= \frac{u^{3/2}}{3/2} + \frac{u^{1/2}}{1/2} + c = \frac{2}{3}u^{3/2} + 2u^{1/2} + c$

Present answer with x's

$= \frac{2}{3}(x-1)^{3/2} + 2(x-1)^{1/2} + c$

Definite integral

(& Substitution method)

eg.:

$$\int_0^1 \frac{3x^2+2}{(x^3+2x)^6}\,dx$$

substitute
write everything in terms of u, including the limits

let $u = x^3+2x$

$\frac{du}{dx} = 3x^2+2$

$du = (3x^2+2)dx$

also change limits

$u = (1)^3+2(1) = 3$

$u = (0)^3+2(0) = 0$

prepare to integrate
– Rewrite

$$\int_0^3 \frac{u}{u^6}\,du = \int_0^3 u^{-5}\,du$$

Integrate
no constant term (+C) with definite integral.

$$= \left[\frac{u^{-4}}{-4}\right]_0^3 = \frac{(3)^{-4}}{-4} - \frac{(0)^{-4}}{-4}$$

evaluate
(use calculator if needed)

$$= -\frac{1}{324}$$

Exponents

3 simple cases

$$\int e^x\,dx = e^x + C$$

eg.1

$$\int e^x\,dx = e^x + c$$

eg.2

$$\int e^{2x}\,dx = \frac{e^{2x}}{2} + c$$

eg.3

$$\int e^{4x+8} = \frac{e^{4x+8}}{4} + c$$

Exponents - with substitution

more complicated example

$$\int_0^1 4xe^{x^2}\,dx$$

Prepare substitution

let $u = x^2$

$\frac{du}{dx} = 2x$

$du = 2x\,dx$

$\Rightarrow 2du = 4x\,dx$

Change limits

$u = (1)^2 = 1$

$u = (0)^2 = 0$

Substitute:

Rewrite interns of u

$$= \int_0^1 2e^u\,du$$

Integrate

$$\int e^x\,dx = e^x + C$$

$$= \left[2e^u\right]_0^1$$

evaluate

$$= 2e^1 - 2e^0 = 2e - 1$$

Integration with natural log (ln) in answer

these may be required

Remember these log Rules

$\ln a + \ln b = \ln ab$

$\ln a - \ln b = \ln \frac{a}{b}$

$\ln a^n = n \ln a$

$$\int \frac{1}{x}\,dx = \ln x + C$$

eg. 1

$$\int \frac{1}{x}\,dx = \ln x + C$$

eg. 2

$$\int \frac{1}{2x}\,dx = \frac{1}{2}\int \frac{1}{x}\,dx = \frac{1}{2}\ln x + C$$

eg.3

$$\int \frac{1}{(5x+4)}\,dx$$

Prepare substitution

let $u = 5x+4$

$\frac{du}{dx} = 5$

$du = 5\,dx$

$\frac{du}{5} = dx$

Substitute: Rewrite

$$\frac{1}{5}\int \frac{1}{u}\,du$$

integrate

$\int \frac{1}{x}dx = \ln x + c$

$$= \frac{1}{5}\ln u + c$$

Rewrite with x's

$$= \frac{1}{5}\ln(5x+4) + C$$

Useful

to

Remember

$\int \frac{f'(x)}{f(x)}\cdot dx$ If Top line is derivative of bottom line!

$= \ln f(x) + c$

* Special Case

eg.4

$$\int \frac{2x+4}{x^2+4x+8}\,dx$$

$$= \ln(x^2+4x+8) + c$$

Knowing this could save time

DIFFERENTIATION

$f(x)$		$f'(x)$
$\sin x$	$\rightarrow$	$\cos x$
$\cos x$	$\rightarrow$	$-\sin x$

INTEGRATION

$\int \sin x\,dx = -\cos x + C$

$\int \cos x\,dx = \sin x + C$

Lesson No. 4: Trigonometric Integration I

2004

8 (a) Find (ii) $\int \cos 6x\,dx$

$= \frac{\sin 6x}{6} + C$

the shape of this answer is worth learning

check: $y = \frac{1}{6}\sin 6x + c$

$\frac{dy}{dx} = \frac{1}{6}(6)\cos 6x = \cos 6x$ ✓

Remembering Differentiation

$y = \cos 3x$ (+c)

Differentiation ↓ ↑ Integration

$\frac{dy}{dx} = -3\sin 3x$

2001

8 (a) Find (ii) $\int \sin 5x\,dx$.

$\int \sin x\,dx = -\cos x + C$

$\int \cos x\,dx = \sin x + C$

$$= -\frac{\cos 5x}{5} + C$$

Check:

$$\frac{dy}{dx} = -\,-\frac{\sin 5x\,(5)}{5}$$

$$= \sin 5x \quad \checkmark$$

2003

8 (c) (i) Show that $\int_a^{2a} \sin 2x\,dx = \sin 3a \sin a$. ?

Integrate

$\int \sin x\,dx = -\cos x + C$

$\int \cos x\,dx = \sin x + C$

evaluate

$$= \left[-\frac{\cos 2x}{2}\right]_a^{2a}$$

$$= -\frac{\cos 2(2a)}{2} + \frac{\cos 2(a)}{2}$$

$$= \frac{1}{2}\left(-\cos 4a + \cos 2a\right)$$

use trig. identity

$\cos A - \cos B$

$= -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$

$$= -\frac{1}{2}\left(\cos 4a - \cos 2a\right) \quad (A = 4a,\ B = 2a)$$

$$= -\frac{1}{2}\left(-2\sin\left(\frac{4a+2a}{2}\right)\sin\left(\frac{4a-2a}{2}\right)\right)$$

$$= \sin 3a \sin a$$

Lesson No. 5: Trigonometric Integration II

2005

8 (b) Evaluate (ii) $\int_0^{\frac{\pi}{8}} \sin^2 2\theta \, d\theta$

Prepare

use trig.

$\sin^2 A = \frac{1}{2}(1 - \cos 2A)$

$\cos^2 A = \frac{1}{2}(1 + \cos 2A)$

$\sin^2 2\theta = \frac{1}{2}(1 - \cos 4\theta) = \frac{1}{2} - \frac{1}{2}\cos 4\theta$

$\int_0^{\pi/8} \left(\frac{1}{2} - \frac{1}{2}\cos 4\theta\right) d\theta$

Integrate

$\int \sin x \, dx = -\cos x + C$

$\int \cos x \, dx = \sin x + C$

$= \left[\frac{1}{2}\theta - \frac{1}{2}\left(\frac{\sin 4\theta}{4}\right)\right]_0^{\pi/8} = \left[\frac{1}{2}\theta - \frac{\sin 4\theta}{8}\right]_0^{\pi/8}$

evaluate

$= \left(\frac{\pi/8}{2} - \frac{\sin 4(\frac{\pi}{8})}{8}\right) - \left(\frac{0}{2} - \frac{\sin(0)}{8}\right)$

$= \left(\frac{\pi}{16} - \frac{1}{8}\right) - (0 - 0) = \frac{\pi - 2}{16}$

2004

Two odd powers

8 (b) Evaluate (ii) $\int_0^{\frac{\pi}{3}} \sin x \cos^3 x \, dx$

Prepare

substitution

Let $u = \cos x$

$\frac{du}{dx} = -\sin x$

$-du = +\sin x \, dx$

Change limits

$u = \cos\left(\frac{\pi}{3}\right) = 1/2$

$u = \cos(0) = 1$

Substitute

$= -\int_1^{\frac{1}{2}} u^3 \, du$

Integrate & evaluate

$\int \sin x \, dx = -\cos x + C$

$\int \cos x \, dx = \sin x + C$

$= -\left[\frac{u^4}{4}\right]_1^{\frac{1}{2}} = -\left[\frac{(\frac{1}{2})^4}{4} - \frac{(1)^4}{4}\right]$

$= +\frac{15}{64}$

Lesson No. 6: Specials

2001

8 (b) Evaluate (i) $\int_0^3 \frac{12}{x^2+9}\,dx$

$\int \frac{dx}{(a)^2+(x)^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c$

$= 12\int_0^3 \frac{dx}{(3)^2+(x)^2}$

$= 4\,\cancel{12}\left[\frac{1}{\cancel{3}}\tan^{-1}\left(\frac{x}{3}\right)\right]_0^3$

$= 4\left[\tan^{-1}\left(\frac{3}{3}\right) - \tan^{-1}\left(\frac{0}{3}\right)\right]$

Degrees $= 4\left[45^\circ - 0^\circ\right] = 180^\circ$ ✓

Radians $= \pi$

2004

8 (b) Evaluate (i) $\int_3^6 \frac{dx}{\sqrt{36-x^2}}$

$\int \frac{dx}{\sqrt{(a)^2-(x)^2}} = \sin^{-1}\left(\frac{x}{a}\right)+c$

$a = 6$

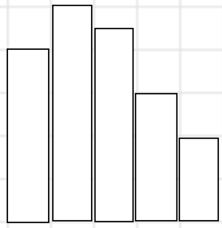
$= \left[\sin^{-1}\left(\frac{x}{6}\right)\right]_3^6$

$= \sin^{-1}\left(\frac{6}{6}\right) - \sin^{-1}\left(\frac{3}{6}\right)$

Degrees $= 90^\circ - 30^\circ = 60^\circ$

Radians $= \pi/3$

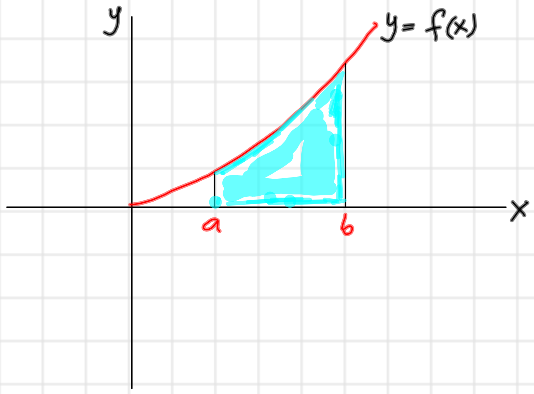
LESSON NO. 7: APPLICATIONS OF INTEGRATION I: AREA



Area Relates to multiplication and to integration

Area between curve and x-axis

$$A = \int_a^b y\, dx$$

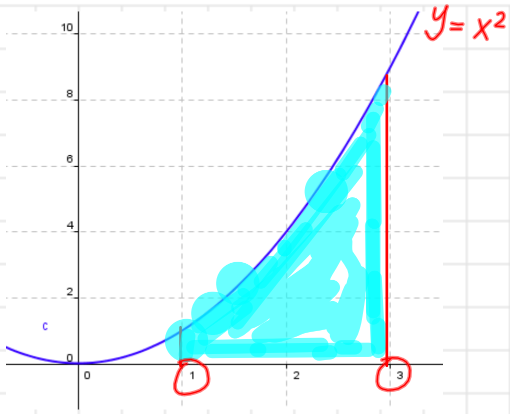


Area

Under

curve

$$A = \int_a^b y\, dx$$



$$A = \int_1^3 x^2\, dx$$

$$\left[\frac{x^3}{3}\right]_1^3 = \left[\frac{(3)^3}{3} - \frac{(1)^3}{3}\right]$$

$$= 9 - \frac{1}{3} = \frac{26}{3}$$

2004

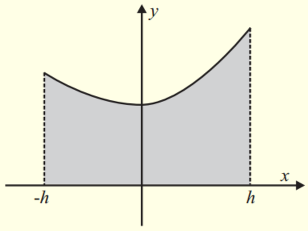
8 (c)

The graph of the function $f(x) = ax^2 + bx + c$ from $x = -h$ to $x = h$ is shown in the diagram.

(i) Show that the area of the shaded region is

$$\tfrac{h}{3}[2ah^2 + 6c].$$

(ii) Given that $f(-h) = y_1$, $f(0) = y_2$ and $f(h) = y_3$, express the area of the shaded region in terms of y_1, y_2, y_3 and h.



$$A = \int_a^b y\, dx$$

integrate

& evaluate

careful with signs

(i)

$$A = \int_{-h}^{h} (ax^2 + bx + c)\, dx$$

$$= \left[\frac{ax^3}{3} + \frac{bx^2}{2} + cx\right]_{-h}^{h}$$

$$= \left(\frac{ah^3}{3} + \frac{bh^2}{2} + ch\right) - \left(-\frac{ah^3}{3} + \frac{bh^2}{2} - ch\right)$$

$$= \frac{2ah^3}{3} + 2ch = \frac{h}{3}\left[2ah^2 + 6c\right]$$ 🙂

2004

8 (c)

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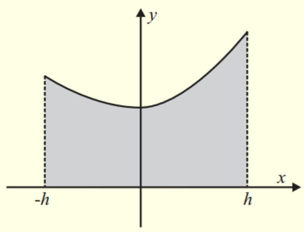


$$A = \frac{h}{3}\left[2ah^2 + 6c\right]$$ 🙂

(ii)

$$f(-h) = a(-h)^2 + b(-h) + c$$

$$= ah^2 - bh + c = y_1$$

$$f(h) = ah^2 + bh + c = y_3$$

$$f(0) = a(0)^2 + b(0) + c = y_2 \Rightarrow c = y_2$$

This is the clever bit.

(for A students!)

$$ah^2 - bh + c = y_1$$

$$+\ ah^2 + bh + c = y_3$$

$$= 2ah^2 + 2c = y_1 + y_3$$

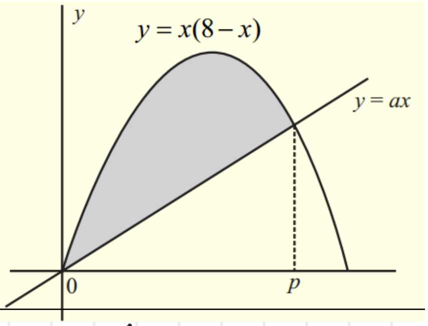
$$A = \frac{h}{3}[y_1 + y_3 + 4c] = \frac{h}{3}[y_1 + y_3 + 4y_2]$$

2001 8 (c)

a is a real number such that $0 < a < 8$.
The line $y = ax$ intersects the curve $y = x(8 - x)$ at $x = 0$ and at $x = p$.

(i) Show that $p = 8 - a$.

(ii) Show that the area between the curve and the line is $\frac{p^3}{6}$ square units.



(i) at intersection $x(8-x) = ax$

divide by X

$\Rightarrow 8 - x = a$ ☺

$A = \int_a^b y\,dx$

(ii) $A = \int_0^p [x(8-x) - ax]\,dx$

note we are integrating curve minus line to get shaded area

$= \int_0^p (8x - x^2 - ax)\,dx = \left[\frac{8x^2}{2} - \frac{x^3}{3} - \frac{ax^2}{2}\right]_0^p$

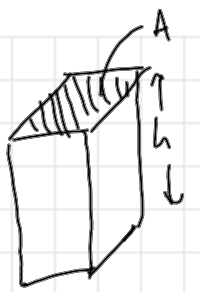
$p = 8 - a$

$\Rightarrow a = 8 - p$

$= \frac{8p^2}{2} - \frac{p^3}{3} - \frac{ap^2}{2} - 0$

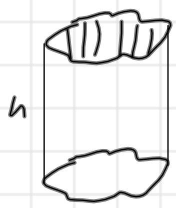
$= \frac{8p^2}{2} - \frac{p^3}{3} - \frac{(8-p)p^2}{2} = \frac{8p^2}{2} - \frac{p^3}{3} - \frac{8p^2}{2} + \frac{p^3}{2} = \frac{p^3}{6}$

Volume



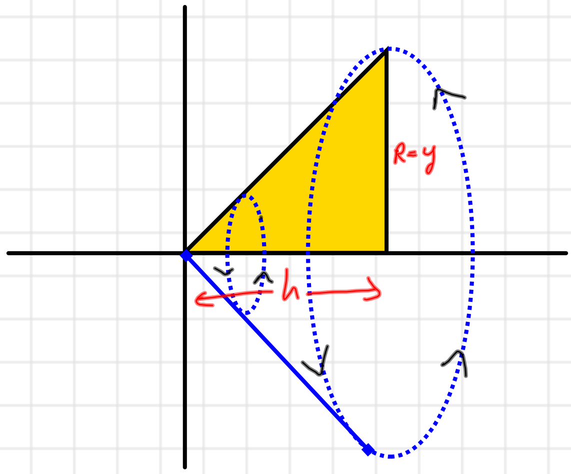
$V = Ah$

$V = Ah$



$V = Ah$

Rotating a line or curve about the x-axis creates a 3D shape



this generates a cone

its volume

$$= \sum \left(\begin{array}{c}\text{Areas} \\ \text{changing} \\ \text{circles}\end{array}\right) dx$$

Sum of

$$\text{Area}_{\text{disc}} = \pi R^2 = \pi y^2$$

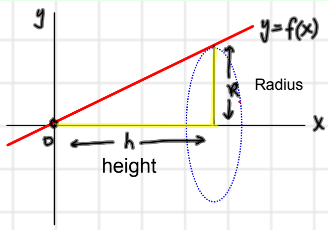
Lesson No. 8: Applications of Integration II: Volume

2005

8 (c) (ii) Use integration methods to derive a formula for the volume of a cone.

First work out the equation of the line

① $y^2 = ?$



Rotate line about x-axis

$y = mx + c$

$m = \frac{\text{Rise}}{\text{Run}} = \frac{R}{h}$

$c = y\text{-intercept} = 0$

$\Rightarrow$ line: $y = \frac{R}{h}x$

② Revolution around the x-axis

$$V = \pi \int_a^b y^2\,dx$$

$$V = \pi \int_0^h \left(\frac{Rx}{h}\right)^2 dx = \pi \int_0^h \frac{R^2x^2}{h^2}$$

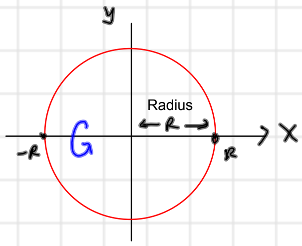
③ Integrate & evaluate

$$= \pi \frac{R^2}{h^2}\left[\frac{x^3}{3}\right]_0^h = \frac{\pi R^2}{h^2}\left[\frac{h^3}{3} - 0\right] = \frac{\pi R^2 h}{3}$$

This is a required derivation

2003

8 (c) (ii) Use integration methods to show that the volume of a sphere with radius r is $\frac{4}{3}\pi r^3$.



Rotate circle about x-axis

Circle with centre (0,0)

$$x^2 + y^2 = R^2$$

$$\Rightarrow y^2 = R^2 - x^2$$

① $y^2 = ?$

② Revolution around the x-axis

$$V = \pi \int_a^b y^2 \, dx$$

③ integrate & evaluate

Careful with signs

$$V = \pi \int_{-R}^{R} (R^2 - x^2)\, dx$$

$$= \pi \left[R^2 x - \frac{x^3}{3} \right]_{-R}^{R}$$

$$= \pi \left[\left(R^2 R - \frac{R^3}{3} \right) - \left(R^2(-R) - \frac{(-R)^3}{3} \right) \right]$$

$$= \pi \left[R^3 - \frac{R^3}{3} + R^3 - \frac{R^3}{3} \right]$$

$$= \pi \left[2R^3 - \frac{2R^3}{3} \right] = \pi \left[\frac{6R^3 - 2R^3}{3} \right] = \frac{4\pi R^3}{3}$$

So now ...