

Leaving Cert Honours Maths

Integration

(Integral Calculus)



Differentiation = Division of changing terms

Integration = Multiplication of changing terms

ADD / SUBTRACT

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

Integration - Introduction

Integration is the opposite to differentiation

We do 3 things

- add 1 to power
- divide by new power
- add C

eg.1 $\int 3x^2 dx = \frac{3x^3}{3} + C = x^3 + C$

eg.2 $\int \frac{1}{x^2} dx = \int x^{-2} = \frac{x^{-1}}{-1} + C = -\frac{1}{x} + C$

eg.3 $\int \frac{1}{\sqrt{x}} dx = \int x^{-\frac{1}{2}} dx = \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + C = 2\sqrt{x} + C$

eg.4 $\int (2x + x^2) dx = \frac{2x^2}{2} + \frac{x^3}{3} + C = x^2 + \frac{x^3}{3} + C$

We do 3 things

- add 1 to power
- divide by new power
- add C

Prepare to integrate

Separate fraction

integrate

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

In these examples we have to 'prepare' them for integration

eg.5 $\int \frac{3+x}{\sqrt{x}} dx$

$\frac{x}{\sqrt{x}} = \frac{\sqrt{x}\sqrt{x}}{\sqrt{x}} = \sqrt{x}$

$$= \int \left(\frac{3}{\sqrt{x}} + \frac{x}{\sqrt{x}} \right) dx = \int (3x^{-1/2} + x^{1/2}) dx$$

$$= \frac{3x^{1/2}}{1/2} + \frac{x^{3/2}}{3/2} + C = 6x^{1/2} + \frac{2}{3}x^{3/2} + C$$

Prepare to integrate

expand

integrate

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

eg.6 $\int \left(x + \frac{1}{x}\right)^2 dx$

$$= \int \left(x^2 + 2(x)\left(\frac{1}{x}\right) + \left(\frac{1}{x}\right)^2\right) dx$$

$$= \int (x^2 + 2 + x^{-2}) dx$$

$$= \frac{x^3}{3} + 2x + \frac{x^{-1}}{-1} + C$$

With limits - Definite Integrals

integrate

no need for +C

evaluate

ie. sub in

top limit

less sub in

lower limit

$$\int_1^2 x^2 dx$$

$$= \left[\frac{x^3}{3} \right]_1^2$$

$$= \frac{(2)^3}{3} - \frac{(1)^3}{3}$$

$$= 4 - \frac{1}{3}$$

$$= 3\frac{2}{3}$$

Multiply, Divide, chain Rule \Rightarrow Substitution Method

eg.1 $\int 2x(x^2+1)^4 dx$

prepare substitution

let $u = x^2 + 1$

$\frac{du}{dx} = 2x$

$du = 2x dx$

Substitute
note: everything must be changed to u's

$-\int u^4 du$

Integrate

$\int x^n dx = \frac{x^{n+1}}{n+1} + C$

$= \frac{u^5}{5} + C$

Present answer with x's

$= \frac{(x^2+1)^5}{5} + C$

Substitution method

eg.2 $\int \frac{x}{\sqrt{x-1}} dx = \int \frac{x}{(x-1)^{1/2}} dx$

prepare substitution

let $u = x - 1$ also $x = u + 1$

$\frac{du}{dx} = 1$

$du = dx$

Substitute

$= \int \frac{u+1}{u^{1/2}} du = \int \left(\frac{u}{u^{1/2}} + \frac{1}{u^{1/2}} \right) du = \int (u^{1/2} + u^{-1/2}) du$

Integrate

$\int x^n dx = \frac{x^{n+1}}{n+1} + C$

$= \frac{u^{3/2}}{3/2} + \frac{u^{1/2}}{1/2} + C = \frac{2}{3} u^{3/2} + 2u^{1/2} + C$

Present answer with x's

$= \frac{2}{3} (x-1)^{3/2} + 2(x-1)^{1/2} + C$

Definite integral

(& Substitution method)

eg.1
$$\int_0^1 \frac{3x^2 + 2}{(x^3 + 2x)^6} dx$$

substitute

write everything in terms of u, including the limits

let $u = x^3 + 2x$
 $\frac{du}{dx} = 3x^2 + 2$
 $du = (3x^2 + 2)dx$

also change limits

$u = (1)^3 + 2(1) = 3$
 $u = (0)^3 + 2(0) = 0$

prepare to integrate - Rewrite

$$\int_0^3 \frac{u}{u^6} du = \int_0^3 u^{-5} du$$

integrate

no constant term (+C) with definite integral.

$$= \left[\frac{u^{-4}}{-4} \right]_0^3 = \frac{(3)^{-4}}{-4} - \frac{(0)^{-4}}{-4}$$

evaluate (use calculator if needed)

$$= -\frac{1}{324}$$

3 simple cases

$$\int e^x dx = e^x + C$$

Exponents

eg.1
$$\int e^x dx = e^x + c$$

eg.2
$$\int e^{2x} dx = \frac{e^{2x}}{2} + c$$

eg.3
$$\int e^{4x+8} = \frac{e^{4x+8}}{4} + c$$

More complicated example

Exponents - with substitution

$$\int_0^1 4xe^{x^2} dx$$

Prepare substitution

$$\begin{aligned} \text{let } u &= x^2 \\ \frac{du}{dx} &= 2x \\ du &= 2x dx \\ \Rightarrow 2du &= 4x dx \end{aligned}$$

Change limits

$$\begin{aligned} u &= (1)^2 = 1 \\ u &= (0)^2 = 0 \end{aligned}$$

Substitute:
Rewrite interns of u

$$= \int_0^1 2e^u du$$

Integrate

$$\int e^x dx = e^x + C$$

$$= [2e^u]_0^1$$

evaluate

$$= 2e^1 - 2e^0 = 2e - 2$$

Integration with natural log (ln) in answer

these may be required

Remember these log Rules

$$\ln a + \ln b = \ln ab$$

$$\ln a - \ln b = \ln \frac{a}{b}$$

$$\ln a^n = n \ln a$$

$$\int \frac{1}{x} dx = \ln x + C$$

eg. 1

$$\int \frac{1}{x} dx = \ln x + C$$

eg. 2

$$\int \frac{1}{2x} dx = \frac{1}{2} \int \frac{1}{x} dx = \frac{1}{2} \ln x + C$$

eg.3

$$\int \frac{1}{(5x+4)} dx$$

Prepare substitution

$$\begin{aligned} \text{let } u &= 5x+4 \\ \frac{du}{dx} &= 5 \\ du &= 5 dx \\ \frac{du}{5} &= dx \end{aligned}$$

Substitute:
Rewrite

$$\frac{1}{5} \int \frac{1}{u} du$$

integrate

$$\int \frac{1}{x} dx = \ln x + c$$

$$= \frac{1}{5} \ln u + c$$

Rewrite with x's

$$= \frac{1}{5} \ln (5x+4) + c$$

Useful
to
Remember

$$\int \frac{f'(x) \cdot dx}{f(x)} \quad \text{if Top line is derivative of bottom line!}$$

$$= \ln f(x) + c$$

* Special case

eg.4

$$\int \frac{2x+4}{x^2+4x+8} dx$$

$$= \ln (x^2+4x+8) + c$$

Knowing this
could save
time

DIFFERENTIATION

$$f(x) \rightarrow f'(x)$$

$$\sin x \rightarrow \cos x$$

$$\cos x \rightarrow -\sin x$$

INTEGRATION

$$\int \sin x dx = -\cos x + C$$

$$\int \cos x dx = \sin x + C$$

LESSON No. 4: TRIGONOMETRIC INTEGRATION I

2004

8 (a) Find (ii) $\int \cos 6x dx$

$$= \frac{\sin 6x}{6} + C$$

*the shape of this answer
is worth learning*

check: $y = \frac{1}{6} \sin 6x + C$

$$\frac{dy}{dx} = \frac{1}{6} (6) \cos 6x = \cos 6x \checkmark$$

Remembering
Differentiation

$$y = \cos 3x \quad (+C)$$

Differentiation \downarrow

$$\frac{dy}{dx} = -3 \sin 3x$$

Integration \uparrow

2001

8 (a) Find (ii) $\int \sin 5x dx$.

$$\int \sin x dx = -\cos x + C$$

$$\int \cos x dx = \sin x + C$$

$$= -\frac{\cos 5x}{5} + C$$

check:

$$\frac{dy}{dx} = -\frac{\sin 5x (5)}{5}$$

$$= \sin 5x \quad \checkmark$$

2003

8 (c) (i) Show that $\int_a^{2a} \sin 2x dx = \underline{\sin 3a \sin a}$.

Integrate

$$\int \sin x dx = -\cos x + C$$

$$\int \cos x dx = \sin x + C$$

evaluate

$$= \left[\frac{-\cos 2x}{2} \right]_a^{2a}$$

$$= \frac{-\cos 2(2a)}{2} + \frac{\cos 2(a)}{2}$$

$$= \frac{1}{2} (-\cos 4a + \cos 2a)$$

$$= -\frac{1}{2} (\cos 4a - \cos 2a)$$

use trig. identity
 $\cos A - \cos B$

$$= -2 \sin\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right)$$

$$= \cancel{-\frac{1}{2}} \left(\cancel{-2} \sin\left(\frac{4a+2a}{2}\right) \sin\left(\frac{4a-2a}{2}\right) \right)$$

$$= \sin 3a \sin a$$

LESSON NO. 5: TRIGONOMETRIC INTEGRATION II

2005

8 (b) Evaluate (ii) $\int_0^{\frac{\pi}{8}} \sin^2 2\theta \, d\theta$

Prepare
use trig.

$\sin^2 A = \frac{1}{2}(1 - \cos 2A)$
 $\cos^2 A = \frac{1}{2}(1 + \cos 2A)$

$\sin^2 2\theta = \frac{1}{2}(1 - \cos 4\theta) = \frac{1}{2} - \frac{1}{2} \cos 4\theta$

Integrate

$\int \sin x \, dx = -\cos x + C$
 $\int \cos x \, dx = \sin x + C$

$\int_0^{\frac{\pi}{8}} \left(\frac{1}{2} - \frac{1}{2} \cos 4\theta\right) d\theta$
 $= \left[\frac{1}{2}\theta - \frac{1}{2} \left(\frac{\sin 4\theta}{4}\right)\right]_0^{\frac{\pi}{8}} = \left[\frac{1}{2}\theta - \frac{\sin 4\theta}{8}\right]_0^{\frac{\pi}{8}}$

evaluate

$= \left(\frac{\pi/8}{2} - \frac{\sin 4(\frac{\pi}{8})}{8}\right) - \left(\frac{0}{2} - \frac{\sin(0)}{8}\right)$
 $= \left(\frac{\pi}{16} - \frac{1}{8}\right) - (0 - 0) = \frac{\pi - 2}{16}$

2004

TWO ODD POWERS

8 (b) Evaluate (ii) $\int_0^{\frac{\pi}{3}} \sin x \cos^3 x \, dx$

Prepare
substitution

let $u = \cos x$	Change limits $u = \cos(\frac{\pi}{3}) = 1/2$ $u = \cos(0) = 1$
$\frac{du}{dx} = -\sin x$	
$-du = +\sin x \, dx$	

Substitute

$= -\int_1^{\frac{1}{2}} u^3 \, du$

Integrate & evaluate
 $\int \sin x \, dx = -\cos x + C$
 $\int \cos x \, dx = \sin x + C$

$= -\left[\frac{u^4}{4}\right]_1^{\frac{1}{2}} = -\left[\frac{(\frac{1}{2})^4}{4} - \frac{(1)^4}{4}\right]$
 $= +\frac{15}{64}$

LESSON NO. 6: SPECIALS

2001

8 (b) Evaluate (i) $\int_0^3 \frac{12}{x^2+9} dx$

$\int \frac{dx}{(a)^2+(x)^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c$

$= 12 \int_0^3 \frac{dx}{(3)^2+(x)^2}$

$= 4 \int_0^3 \left[\frac{1}{3} \tan^{-1}\left(\frac{x}{3}\right) \right] dx$

$= 4 \left[\tan^{-1}\left(\frac{3}{3}\right) - \tan^{-1}\left(\frac{0}{3}\right) \right]$

$= 4 \left[45^\circ - 0^\circ \right] = 180^\circ \checkmark$

$= \pi$

Degrees

Radians

2004

8 (b) Evaluate (i) $\int_3^6 \frac{dx}{\sqrt{36-x^2}}$

$\int \frac{dx}{\sqrt{(a)^2-(x)^2}} = \sin^{-1}\left(\frac{x}{a}\right) + c$

$a = 6$

$= \left[\sin^{-1}\left(\frac{x}{6}\right) \right]_3^6$

$= \sin^{-1}\left(\frac{6}{6}\right) - \sin^{-1}\left(\frac{3}{6}\right)$

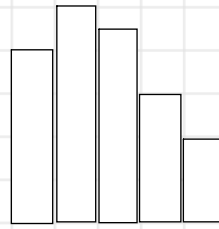
$= 90^\circ - 30^\circ = 60^\circ$

$= \pi/3$

Degrees

Radians

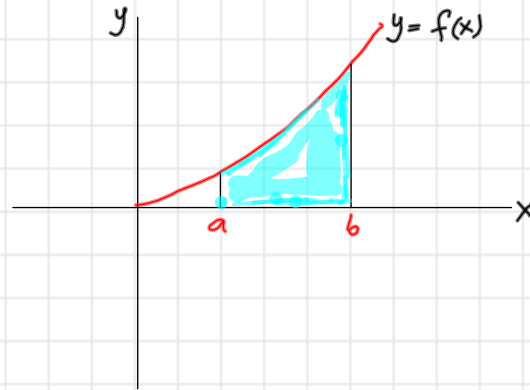
LESSON NO. 7: APPLICATIONS OF INTEGRATION I: AREA



Area relates to multiplication and to integration

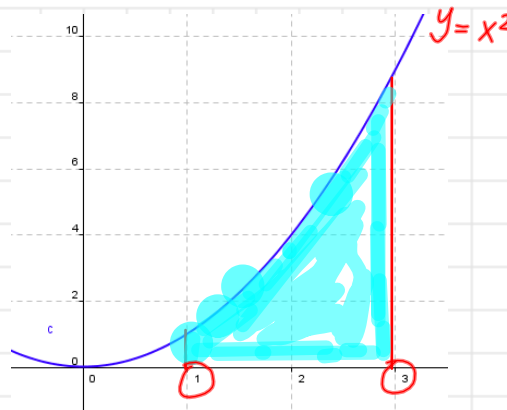
Area between curve and x-axis

$$A = \int_a^b y \, dx$$



Area Under curve

$$\Delta = \int_a^b y \, dx$$



$$\Delta = \int_1^3 x^2 \, dx$$

$$\left[\frac{x^3}{3} \right]_1^3 = \left[\frac{(3)^3}{3} - \frac{(1)^3}{3} \right]$$

$$= 9 - \frac{1}{3} = \frac{26}{3}$$

2004

8 (c)

$$A = \int_a^b y \, dx$$

integrate
& evaluate

careful with signs

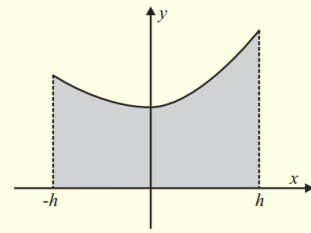
The graph of the function $f(x) = ax^2 + bx + c$ from $x = -h$ to $x = h$ is shown in the diagram.

(i) Show that the area of the shaded region is

$$\frac{h}{3}[2ah^2 + 6c].$$

(ii) Given that $f(-h) = y_1$, $f(0) = y_2$ and

$f(h) = y_3$, express the area of the shaded region in terms of y_1 , y_2 , y_3 and h .



$$\begin{aligned} \text{(i)} \quad A &= \int_{-h}^h (ax^2 + bx + c) \, dx \\ &= \left[\frac{ax^3}{3} + \frac{bx^2}{2} + cx \right]_{-h}^h \\ &= \left(\frac{ah^3}{3} + \frac{bh^2}{2} + ch \right) - \left(-\frac{ah^3}{3} + \frac{bh^2}{2} - ch \right) \\ &= \frac{2ah^3}{3} + 2ch = \frac{h}{3} [2ah^2 + 6c] \quad \text{😊} \end{aligned}$$

2004

8 (c)

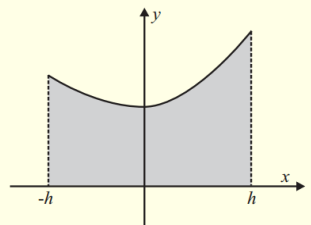
The graph of the function $f(x) = ax^2 + bx + c$ from $x = -h$ to $x = h$ is shown in the diagram.

(i) Show that the area of the shaded region is

$$\frac{h}{3}[2ah^2 + 6c].$$

(ii) Given that $f(-h) = y_1$, $f(0) = y_2$ and

$f(h) = y_3$, express the area of the shaded region in terms of y_1 , y_2 , y_3 and h .



$$\begin{aligned} A &= \frac{h}{3} [2ah^2 + 6c] \quad \text{😊} \\ \text{(ii)} \quad f(-h) &= a(-h)^2 + b(-h) + c \\ &= ah^2 - bh + c = y_1 \\ f(h) &= ah^2 + bh + c = y_3 \\ f(0) &= a(0)^2 + b(0) + c = y_2 \Rightarrow c = y_2 \\ ah^2 - bh + c &= y_1 \\ + ah^2 + bh + c &= y_3 \\ \hline &= 2ah^2 + 2c = y_1 + y_3 \\ A &= \frac{h}{3} [y_1 + y_3 + 4c] = \frac{h}{3} [y_1 + y_3 + 4y_2] \end{aligned}$$

This is the clever bit.

(for A students!)

2001

a is a real number such that $0 < a < 8$.

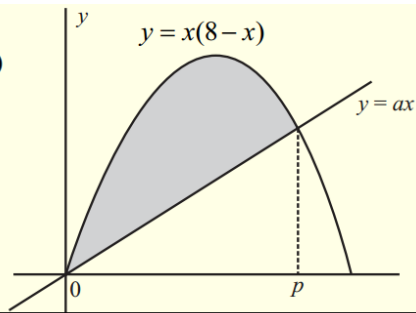
8 (c)

The line $y = ax$ intersects the curve $y = x(8-x)$ at $x = 0$ and at $x = p$.

(i) Show that $p = 8 - a$.

(ii) Show that the area between the curve

and the line is $\frac{p^3}{6}$ square units.



divide by X

$$A = \int_a^b y \, dx$$

note we are integrating curve minus line to get shaded area

$$p = 8 - a$$

$$\Rightarrow a = 8 - p$$

(i) at intersection $x(8-x) = ax$
 $\Rightarrow 8-x = a$ 😊

(ii)

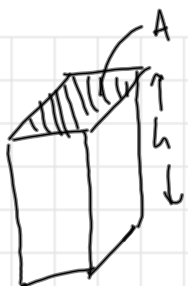
$$A = \int_0^p [x(8-x) - ax] \, dx$$

$$= \int_0^p (8x - x^2 - ax) \, dx = \left[\frac{8x^2}{2} - \frac{x^3}{3} - \frac{ax^2}{2} \right]_0^p$$

$$= \frac{8p^2}{2} - \frac{p^3}{3} - \frac{ap^2}{2} - 0$$

$$= \frac{8p^2}{2} - \frac{p^3}{3} - \frac{(8-p)p^2}{2} = \frac{8p^2}{2} - \frac{p^3}{3} - \frac{8p^2}{2} + \frac{p^3}{2} = \frac{p^3}{6}$$

Volume



$$V = Ah$$



$$V = Ah$$



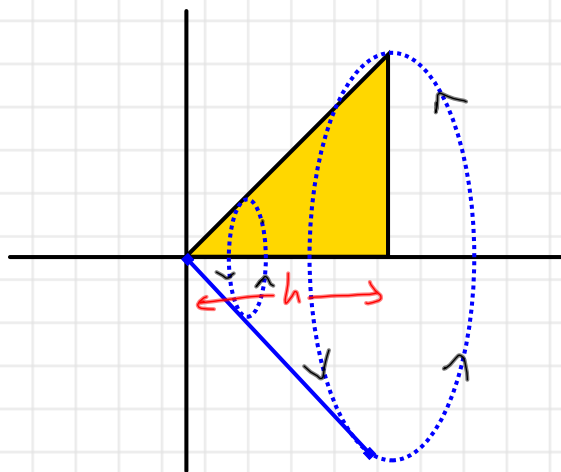
$$V = Ah$$

Rotating a line OR curve about the x-axis creates a 3D shape

this generates a cone

Its Volume

$$= \left(\text{Areas Changing circles} \right) dx$$



Area = πR^2
disc

LESSON NO. 8: APPLICATIONS OF INTEGRATION II: VOLUME

2005

8 (c) (ii) Use integration methods to derive a formula for the volume of a cone.

First work out the equation of the line

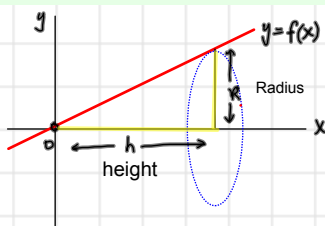
① $y^2 = ?$

②

Revolution around the x-axis
a 2
b

③

Integrate & evaluate



Rotate line about x-axis

$$y = mx + c$$

$$m = \frac{\text{Rise}}{\text{Run}} = \frac{R}{h}$$

$$c = \text{y-intercept} = 0$$

$$\Rightarrow \text{line: } y = \frac{R}{h} x$$

$$V = \pi \int_0^h \left(\frac{Rx}{h} \right)^2 dx = \pi \int_0^h \frac{R^2 x^2}{h^2}$$

$$= \pi \frac{R^2}{h^2} \left[\frac{x^3}{3} \right]_0^h = \pi \frac{R^2}{h^2} \left[\frac{h^3}{3} - 0 \right] = \frac{\pi R^2 h}{3} \text{ 😊}$$

This is a required derivation

2003

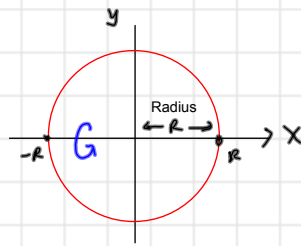
8 (c) (ii) Use integration methods to show that the volume of a sphere with radius r is $\frac{4}{3}\pi r^3$.

① $y^2 = ?$

② Revolution around the x-axis
a 2
b

③ integrate & evaluate

Careful with signs



Rotate circle about x-axis

Circle with centre $(0,0)$

$$x^2 + y^2 = R^2$$

$$\Rightarrow y^2 = R^2 - x^2$$

$$V = \pi \int_{-R}^R (R^2 - x^2) dx$$

$$= \pi \left[R^2 x - \frac{x^3}{3} \right]_{-R}^R$$

$$= \pi \left[\left(R^2 R - \frac{R^3}{3} \right) - \left(R^2 (-R) - \frac{(-R)^3}{3} \right) \right]$$

$$= \pi \left[R^3 - \frac{R^3}{3} + R^3 - \frac{R^3}{3} \right]$$

$$= \pi \left[2R^3 - \frac{2R^3}{3} \right] = \pi \left[\frac{6R^3 - 2R^3}{3} \right] = \frac{4\pi R^3}{3} \quad \text{😊}$$