Proof By Induction

Mathematical induction is a method of mathematical proof typically used to establish that a given statement is true for all natural numbers (positive integers). It is done by proving that the **first** statement in the infinite sequence of statements is true, and then proving that if **any one** statement in the infinite sequence of statements is true, then so is the **next** one.

Step 1: Prove that the proposition is true for the smallest value of n given in the question, usually $\boldsymbol{n = 1}$
Step 2: Assume the proposition is true for ***n=k***
Step 3: Show that the proposition is true for ***n=k+1***
Step 4: The proposition is true for $n = 1$. If the proposition is true for $n = k$, then it will be true for $n = k + 1$. Therefore by induction it is true for all $n \in N$

We use it in 3 areas: Divisibility, Series and Inequalities

Divisibility

Prove by induction that 8 is a factor $7^{2n+1} + 1$ for $n \in N$

Step 1: Show true for $n = 1$

$$7^{2(1)+1} + 1 = 7^3 + 1 = 344$$

which is divisible by 8

Step 2: Assume true for $n = k$

$$7^{2k+1} + 1$$

is divisible by 8

Step 3: Prove true for $n = k + 1$
To prove:

$$7^{2(k+1)+1} + 1 = 7^{2k+3} + 1$$

is divisible by 8

$$f(k+1) - f(k) = 7^{2k+3} + 1 - (7^{2k+1} + 1)$$

$$= 7^{2k}.7^3 + 1 - 7^{2k}.7^1 - 1$$
$$= 7^{2k}(7^3 - 7^1)$$
$$= 7^{2k}(336)$$

which is divisible by 8

The proposition is true for $n = 1$. If the proposition is true for $n = k$, then it will be true for $n = k + 1$. Therefore by induction it is true for all $n \in N$

Series

Prove by induction that the sum of the first n natural numbers $\sum n = \frac{n(n+1)}{2}$ for $n \in N$.

Step 1: Show true for $n = 1$

$$1 = \frac{1(1+1)}{2}$$
$$1 = 1$$

Step 2: Assume true for $n = k$

$$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}$$

Step 3: Prove true for $n = k + 1$
To prove:

$$1 + 2 + 3 + \cdots + k + k + 1 = \frac{(k+1)(k+2)}{2}$$

Proof:

$$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2} \textit{Assumption}$$

$$1 + 2 + 3 + \cdots + k + k + 1 = \frac{k(k+1)}{2} + k + 1$$
$$= \frac{k^2 + k}{2} + k + 1$$
$$= \frac{k^2 + k + 2k + 2}{2}$$
$$= \frac{k^2 + 3k + 2}{2}$$
$$= \frac{(k+1)(k+2)}{2}$$

The proposition is true for $n = 1$. If the proposition is true for $n = k$, then it will be true for $n = k + 1$. Therefore by induction it is true for all $n \in N$

Inequalities

Prove by induction that $2^n \geq n^2$ for $n \geq 4, n \in N$

Step 1: Show true for $n = 4$

$$2^4 \geq 4^2$$

Step 2: Assume true for $n = k$

$$2^k \geq k^2$$

Step 3: Prove true for $n = k + 1$
To prove:

$$2^{k+1} \geq (k+1)^2$$

Since $2^k \geq k^2$
$2^k.2 \geq 2k^2$
Therefore
$2^{k+1} \geq 2k^2$
So we need to show that $2k^2 \geq (k+1)^2$

$$2k^2 \geq k^2 + 2k + 1$$
$$k^2 - 2k - 1 \geq 0$$
$$k^2 - 2k + 1 - 2 \geq 0$$
$$(k-1)^2 - 2 \geq 0$$

Which is true for $k \geq 4$

$$2^{k+1} \geq 2k^2 \geq (k+1)^2$$

The proposition is true for $n = 1$. If the proposition is true for $n = k$, then it will be true for $n = k + 1$. Therefore by induction it is true for all $n \in N$