Proofs

The following are the proofs that must be learned off by heart for Higher Level Maths. Whilst some of them can be worked out you are better off knowing them really well so that they will be easy marks on the day and free up time elsewhere.

As well as knowing them off by heart you need to know how to use them as they can be asked in other ways other than a straight proof recitation.

Complex Numbers

- De Moivre's Theorem
- De Moivre's Theorem to Prove Trigonometric Identities

Differentiation

- 1^{st} Principles – x^2
- 1^{st} Principles – x^3
- 1^{st} Principles – $\sqrt{x}$
- 1^{st} Principles – $\frac{1}{x}$
- 1^{st} Principles – $\cos x$
- 1^{st} Principles – $\sin x$
- General Rule
- Sum Rule
- Product Rule
- Quotient Rule

Integration

- Volume of Cone
- Volume of Sphere

Proof By Contradiction

- Proof that $\sqrt{2}$ is irrational

Trigonometry

- $\cos^2 A + \sin^2 A = 1$
- $\frac{a}{sinA} = \frac{b}{sinB} = \frac{c}{sinC}$
- $b^2 = a^2 + c^2 - 2ac\cos B$
- $\cos(A - B) = \cos A \cos B + \sin A \sin B$
- $\cos(A + B) = \cos A \cos B - \sin A \sin B$
- $\cos 2A = \cos^2 A - \sin^2 A$
- $\sin(A + B) = \sin A \cos B + \cos A \sin B$
- $\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

Complex Numbers

De Moivre's Theorm

$(\boldsymbol{\cos\theta + i\sin\theta})^{\boldsymbol{n}} = \boldsymbol{\cos n\theta + i\sin n\theta}$

Proof by Induction

1. Show true for $n = 1$

Proof $(\cos\theta + i\sin\theta)^1 = \cos 1\theta + i\sin 1\theta$

This is obviously true.

2. Assume true for $n = k$

Assume $(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$

3. Prove true for $n = k + 1$

Prove $(\cos\theta + i\sin\theta)^{k+1} = \cos(k+1)\theta + i\sin(k+1)\theta$

Proof:

$(\cos\theta + i\sin\theta)^{k+1} = (\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta)^1$

$= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$

$= (\cos k\theta\cos\theta - \sin k\theta\sin\theta) + i(\sin k\theta\cos\theta + \cos k\theta\sin\theta)$

$= \cos(k\theta + \theta) + i\sin(k\theta + \theta)$

$= \cos(k+1)\theta + i\sin(k+1)\,\theta$

Therefore it is true for $n = k$ true fro $n = k + 1$

So true for $n = 1$ and $n = k + 1$ true for all $n \in N$

De Moivre for Negative Integers

$z = \cos\theta + i\sin\theta$

Prove that $\boldsymbol{z^{-n} = \cos n\theta - i\sin n\theta}$

$z^n = \cos n\theta\ + i\sin n\theta$

$\frac{1}{z^n} = \frac{1}{(\cos\theta\ + i\sin\theta)^n}$

$z^{-n} = (\cos\theta\ + i\sin\theta)^{-n}$

$z^{-n} = \cos(-n\theta) + i\sin(-n\theta)$

$z^{-n} = \cos n\theta - i\sin n\theta$

De Moivre's Theorem to Prove Trigonometric Identities

We can use De Moivre to prove trigonometric identities by expressing $\cos n\theta$ and $\sin n\theta$ as polynomials in $\cos\theta$ and $\sin\theta$

Example 1 –

Using De Moivre's Theorem prove that $cos3\theta = 4cos^3\theta - 3cos\theta$

Solution

De Moivre's Theorem $(cos\theta + isin\theta)^n = cos\ n\theta + isin\ n\theta$

Therefore by De Moivre's Theorem
$cos3\theta + isin3\theta = (cos\theta + isin\theta)^3$

Binomial Expansion (or multiply out long way)

$cos3\theta + isin3\theta = cos^3\theta + 3cos^2\theta(isin\theta) + 3cos\theta(i^2sin^2\theta) + i^3sin^3\theta$
$cos3\theta + isin3\theta = cos^3\theta + 3cos^2\theta(isin\theta) - 3cos\theta sin^2\theta - isin^3\theta$
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Equating Real Parts

$cos3\theta = cos^3\theta - 3cos\theta sin^2\theta$
$cos3\theta = cos^3\theta - 3cos\theta(1 - cos^2\theta)$
$cos3\theta = cos^3\theta - 3cos\theta + 3cos^3\theta$
$cos3\theta = 4cos^3\theta - 3cos\theta$

Differentiation

1st principles

$\boldsymbol{f(x) = x^2}$

$f(x+h) = (x+h)^2$

$f(x+h) - f(x) = (x+h)^2 - x^2$

$f(x+h) - f(x) = x^2 + 2xh + h^2 - x^2$

$f(x+h) - f(x) = 2xh + h^2$

$$\frac{f(x+h) - f(x)}{h} = 2x + h$$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = 2x$$

$\boldsymbol{f(x) = x^3}$

$f(x+h) = (x+h)^3$

$f(x+h) - f(x) = (x+h)^3 - x^3$

$f(x+h) - f(x) = x^3 + 3x^2h + 3xh^2 + h^3 - x^3$

$f(x+h) - f(x) = 3x^2h + 3xh^2 + h^3$

$$\frac{f(x+h) - f(x)}{h} = 3x^2 + 3xh + h^2$$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = 3x^2$$

$\boldsymbol{f(x) = \frac{1}{x}}$

$$f(x+h) = \frac{1}{x+h}$$

$$f(x+h) - f(x) = \frac{1}{x+h} - \frac{1}{x}$$

$$f(x+h) - f(x) = \frac{x - (x+h)}{(x+h)(x)}$$

$$f(x+h) - f(x) = \frac{-h}{x^2 + h}$$

$$\frac{f(x+h) - f(x)}{h} = \frac{-1}{x^2 + h}$$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = -\frac{1}{x^2}$$

$\boldsymbol{f(x) = \sqrt{x}}$

$f(x+h) = \sqrt{x+h}$

$f(x+h) - f(x) = \sqrt{x+h} - \sqrt{x}$

$$f(x+h) - f(x) = \sqrt{x+h} - \sqrt{x} \times \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$f(x+h) - f(x) = \frac{x + h - x}{\sqrt{x+h} + \sqrt{x}}$$

$$f(x+h) - f(x) = \frac{h}{\sqrt{x+h} + \sqrt{x}}$$

$$\frac{f(x+h) - f(x)}{h} = \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

$\boldsymbol{f(x) = \sin x}$

$f(x+h) = \sin(x+h)$

$f(x+h) - f(x) = \sin(x+h) - \sin x$

$f(x+h) - f(x) = 2\cos\left(\frac{x+h+x}{2}\right)\sin\left(\frac{x+h-x}{2}\right)$

$f(x+h) - f(x) = 2\cos\left(\frac{2x+h}{2}\right)\sin\left(\frac{h}{2}\right)$

$\frac{f(x+h) - f(x)}{h} = \frac{2\cos\left(\frac{2x+h}{2}\right)\sin\left(\frac{h}{2}\right)}{h}$

$\frac{f(x+h) - f(x)}{h} = 2\cos\left(\frac{2x+h}{2}\right)\frac{\sin\left(\frac{h}{2}\right)}{h}$

$\frac{f(x+h) - f(x)}{h} = 2\cos\left(\frac{2x+h}{2}\right)\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}.\frac{1}{2}$

$\lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = 2\cos\left(\frac{2x}{2}\right).1.\frac{1}{2}$

$= \cos x$

$\boldsymbol{f(x) = \cos x}$

$f(x+h) = \cos(x+h)$

$f(x+h) - f(x) = \cos(x+h) - \cos x$

$f(x+h) - f(x) = -2\sin\left(\frac{x+h+x}{2}\right)\sin\left(\frac{x+h-x}{2}\right)$

$f(x+h) - f(x) = -2\sin\left(\frac{2x+h}{2}\right)\sin\left(\frac{h}{2}\right)$

$\frac{f(x+h) - f(x)}{h} = \frac{-2\sin\left(\frac{2x+h}{2}\right)\sin\left(\frac{h}{2}\right)}{h}$

$\frac{f(x+h) - f(x)}{h} = -2\sin\left(\frac{2x+h}{2}\right)\frac{\sin\left(\frac{h}{2}\right)}{h}$

$\frac{f(x+h) - f(x)}{h} = -2\sin\left(\frac{2x+h}{2}\right)\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}.\frac{1}{2}$

$\lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = -2\sin\left(\frac{2x}{2}\right).1.\frac{1}{2}$

$= -\sin x$

General Rule

If $y = x^n$ prove that $\frac{dy}{dx} = nx^{n-1}$

Proof by induction

Prove for $n = 1$ using 1st Principles

$f(x) = x$

$f(x+h) = x + h$

$f(x+h) - f(x) = x + h - x$

$f(x+h) - f(x) = h$

$$\frac{f(x+h) - f(x)}{h} = 1$$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = 1$$

Assume true for $n = k$

$y = x^k$ prove that $\frac{dy}{dx} = kx^{k-1}$

Prove for $n = k + 1$

$y = x^{k+1}$ then $\frac{dy}{dx} = (k+1)x^k$

$y = x^{k+1} = x.x^k$

Use product rule

$$\frac{dy}{dx} = 1.x^k + x.kx^{k-1}$$

$= x^k + kx^k$

$= (k+1)x^k$

True for $n = 1$, proven true for $n = k + 1$ then true in general

Sum Rule

$f(x) = u(x) + v(x)$

$f(x+h) = u(x+h) + v(x+h)$

$f(x+h) - f(x) = u(x+h) + v(x+h) - \left(u(x) + v(x)\right)$

$f(x+h) - f(x) = u(x+h) - u(x) + v(x+h) - v(x)$

$$\frac{f(x+h) - f(x)}{h} = \frac{u(x+h) - u(x) + v(x+h) - v(x)}{h}$$

$$\frac{f(x+h) - f(x)}{h} = \frac{u(x+h) - u(x)}{h} + \frac{v(x+h) - v(x)}{h}$$

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \frac{du}{dx} + \frac{dv}{dx}$$

Product Rule

$$f(x) = u(x)v(x)$$

$$f(x+h) = u(x+h)v(x+h)$$

$$f(x+h) - f(x) = u(x+h)v(x+h) - u(x)v(x)$$

$$f(x+h) - f(x) = u(x+h)v(x+h) - u(x)v(x+h) + u(x)v(x+h) - u(x)v(x)$$

$$f(x+h) - f(x) = v(x+h)[u(x+h) - u(x)] + u(x)[v(x+h) - v(x)]$$

$$\frac{f(x+h) - f(x)}{h} = \frac{v(x+h)[u(x+h) - u(x)] + u(x)[v(x+h) - v(x)]}{h}$$

$$\frac{f(x+h) - f(x)}{h} = v(x+h)\frac{u(x+h) - u(x)}{h} + u(x)\frac{v(x+h) - v(x)}{h}$$

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = v(x)\frac{du}{dx} + u(x)\frac{dv}{dx}$$

Quotient Rule

$$f(x) = \frac{u(x)}{v(x)}$$

$$f(x+h) = \frac{u(x+h)}{v(x+h)}$$

$$f(x+h) - f(x) = \frac{u(x+h)}{v(x+h)} - \frac{u(x)}{v(x)}$$

$$f(x+h) - f(x) = \frac{u(x+h)v(x) - u(x)v(x+h)}{v(x+h)v(x)}$$

$$f(x+h) - f(x) = \frac{u(x+h)v(x) - u(x)v(x) + u(x)v(x) - u(x)v(x+h)}{v(x+h)v(x)}$$

$$f(x+h) - f(x) = \frac{v(x)[u(x+h) - u(x)] - u(x)[v(x+h) - v(x)]}{v(x+h)v(x)}$$

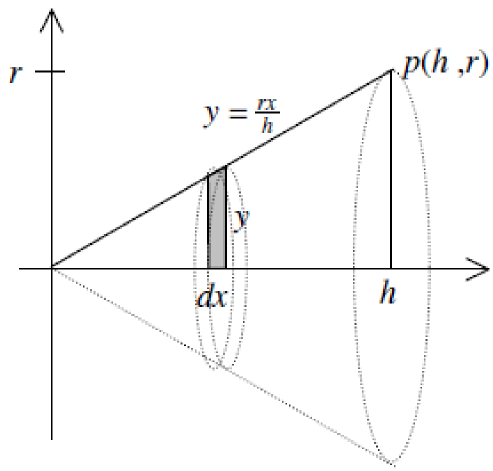
$$\frac{f(x+h) - f(x)}{h} = \frac{\frac{v(x)[u(x+h) - u(x)] - u(x)[v(x+h) - v(x)]}{h}}{v(x+h)v(x)}$$

$$\frac{f(x+h) - f(x)}{h} = \frac{v(x)\frac{u(x+h) - u(x)}{h} - u(x)\frac{v(x+h) - v(x)}{h}}{v(x+h)v(x)}$$

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \frac{v(x)\frac{du}{dx} - u(x)\frac{dv}{dx}}{v^2(x)}$$

Integration

Prove using integration methods that the volume of a cone is $\frac{1}{3}\pi r^2 h$



Draw a line through the origin with a slope of $\frac{r}{h}$

$$V_x = \pi \int_a^b y^2 dx$$

$$V_x = \pi \int_0^h \frac{r^2}{h^2} x^2 dx$$

$$V_x = \frac{\pi r^2}{h^2} \int_0^h x^2 dx$$

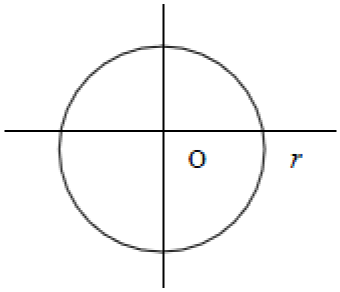
$$V_x = \frac{\pi r^2}{h^2} \left[\frac{x^3}{3}\right]$$

$$V_x = \frac{\pi r^2}{h^2} \left[\frac{(h)^3}{3} - \frac{(0)^3}{3}\right]$$

$$V_x = \frac{\pi r^2}{h^2} \left[\frac{h^3}{3}\right]$$

$$V_x = \frac{1}{3} \pi r^2 h$$

Prove using integration methods that the volume of a sphere is $\frac{4}{3}\pi r^3$



A circle centre (0,0) is given by $x^2 + y^2 = r^2$

$$V = \pi \int_a^b y^2 dx$$

Find the volume of half a circle from 0 to r

$$y^2 = r^2 - x^2$$

$$V = \pi \int_0^r (r^2 - x^2) dx$$

$$V = \left[r^2 x - \frac{x^3}{3} \right]_0^r$$

$$V = r^2(r) - \frac{(r)^3}{3} - r^2(0) + \frac{(0)^3}{3}$$

$$V = r^3 - \frac{r^3}{3}$$

$$V = \frac{2}{3}\pi r^3$$

Multiply by 2 to get the volume of sphere

$$V = \frac{4}{3}\pi r^3$$

Proof $\sqrt{2}$ is irrational

Proof that $\sqrt{2}$ is irrational, that it cannot be written $\frac{p}{q}$ where p and q are integers.

Suppose that $\sqrt{2}$ is irrational, that it **can** be written as a fraction $\frac{p}{q}$ where p and q are integers.

$$\left(\frac{p}{q}\right) = \sqrt{2}$$

Square both sides

$$\left(\frac{p}{q}\right)^2 = 2$$

Multiply both sides by q^2

$$p^2 = 2q^2$$

If $p^2 = 2q^2$, then we know we have reached a contradiction.

So 2 is a factor of p^2

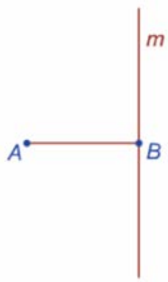
The argument is correct, so the assumption on which it was based must be false.

Hence $\sqrt{2}$ cannot be written as a fraction $\frac{p}{q}$ where p and q are integers.

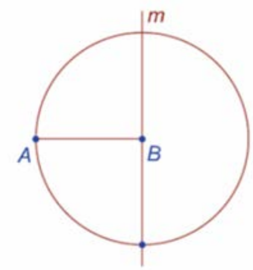
Construct $\sqrt{2}$

1. Let the line segment AB be of length 1 unit.

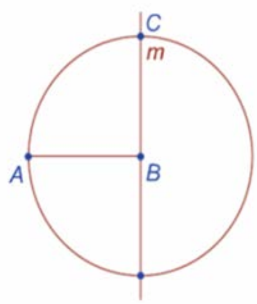
2. Construct a line m perpendicular to $[AB]$ at B.



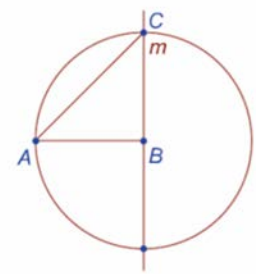
3. Construct a circle with centre B and radius $[AB]$.



4. Mark the intersection, C, of the circle and m.



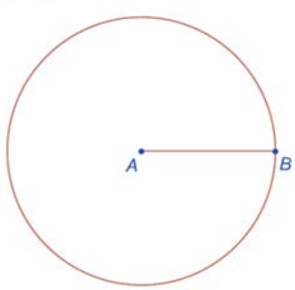
5. Draw the line segment CA.
$|AC| = \sqrt{2}$



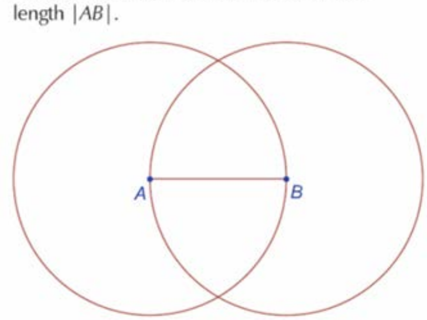
Construct $\sqrt{3}$

1. Let the line segment AB be of length 1 unit.

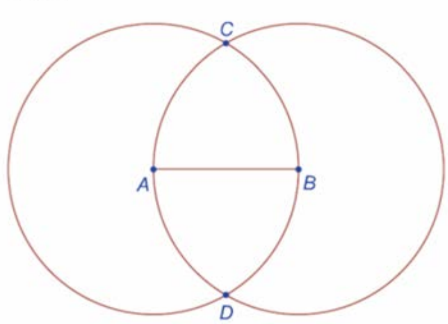
2. Construct a circle with centre A and radius length $|AB|$.



3. Construct a circle with centre B and radius length $|AB|$.



4. Mark the intersection of the two circles as C and D.



5. Draw the line segment $[CD]$.
$|CD| = \sqrt{3}$