Trigonometric Proofs

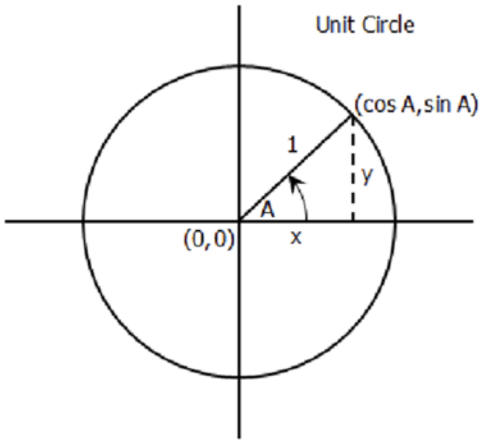
Proofs 1 – 7 and 9 required for Higher Level

$\cos^2 A + \sin^2 A = 1$

Distance from (0, 0) to (cos A, sin A) is 1

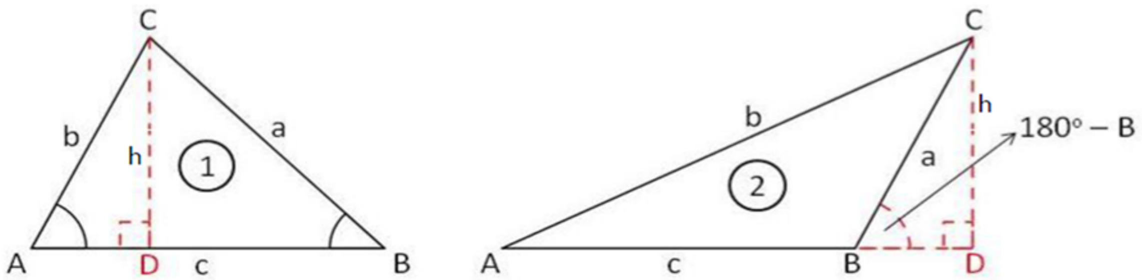
$\Rightarrow \sqrt{(\cos A - 0)^2 + \sin(A-0)^2} = 1$

$\Rightarrow \cos^2 A + \sin^2 A = 1$



$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

Need to examine two cases – acute angled triangles such as ΔACB and obtuse angled triangles such as ΔBCD



Case 1: ΔACB (acute angled)

$\sin A = \frac{h}{b} \Rightarrow h = b\sin A$

$\sin B = \frac{h}{a} \Rightarrow h = a\sin B$

Case 2: ΔBCD (obtuse angled)

$\sin A = \frac{h}{b} \Rightarrow h = b\sin A$

$\sin(180 - B) = \frac{h}{a}$

$\sin B = \frac{h}{a}$ [as $\sin(180 - B) = \sin B$]

$\Rightarrow h = a\sin B$

In both cases: h= bsinA and h = asinB

Equating h's

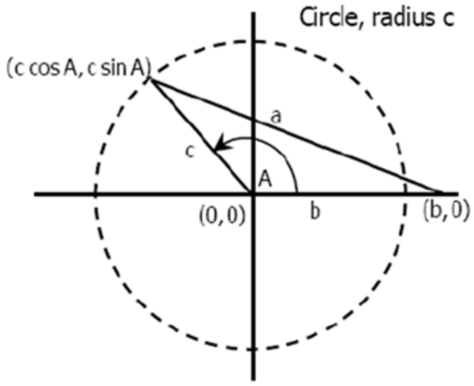
$a\sin B = b\sin A \Rightarrow \frac{a}{\sin A} = \frac{b}{\sin B}$

Similarly if the perpendicular height was dropped from A it would yield:

$\frac{b}{\sin B} = \frac{c}{\sin C}$

$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

$a^2 = b + c - 2bc\cos A$



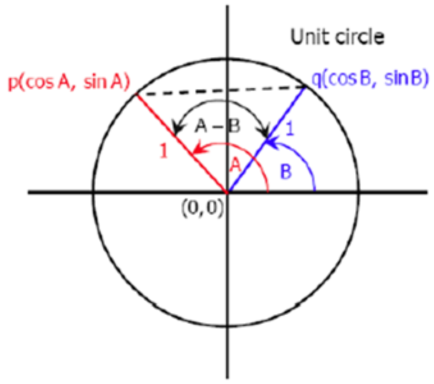
$a = \sqrt{(c\cos A - b)^2 + (c\sin A - 0)^2}$ using the distance formula

$a^2 = c^2\cos^2 A - 2bc\cos A + b^2 + c^2\sin^2 A$

$a^2 = b^2 + c^2(\cos^2 A + \sin^2 A) - 2bc\cos A$

$a^2 = b^2 + c^2 - 2bc\cos A$ as $\cos^2 A + \sin^2 A = 1$

$\cos(A - B) = \cos A\cos B + \sin A\sin B$



Find the distance between p and q in two different ways and equate the anwsers

$|pq|^2 = 1^2 + 1^2 - 2(1)(1)\cos(A - B)$ using cosine formula, $a^2 = b^2 + c^2 - 2bc\cos A$

$|pq|^2 = 2 - 2\cos(A - B)$

$|pq| = \sqrt{(\cos A - \cos B)^2 + (\sin A - \sin B)^2}$ using distance formula

$|pq|^2 = \cos^2 A - 2\cos A\cos B + \cos^2 B + \sin^2 A - 2\sin A\sin B + \sin^2 B$

$|pq|^2 = 2 - 2\cos A\cos B - 2\sin A\sin B$

Equating both:

$2 - 2\cos(A - B) = 2 - 2\cos A\cos B - 2\sin A\sin B$

$-2\cos(A - B) = -2\cos A\cos B - 2\sin A\sin B$

$\cos(A - B) = \cos A\cos B + \sin A\sin B$

$\cos(A+B) = \cos A \cos B - \sin A \sin B$

$\cos(A-B) = \cos A \cos B + \sin A \sin B$ — using formula 4

$\cos(A-(-B)) = \cos A \cos(-B) + \sin A \sin(-B)$ — changing B to $-B$

$\cos(A+B) = \cos A \cos B - \sin A \sin B$ — as $\cos(-B) = \cos B$ and $\sin(-B) = -\sin B$ [pg. 13]

$\cos 2A = \cos^2 A - \sin^2 A$

$\cos(A+B) = \cos A \cos B - \sin A \sin B$ — using formula 5

$\cos(A+A) = \cos A \cos A - \sin A \sin A$ — changing B to A

$\cos 2A = \cos^2 A - \sin^2 A$

$\sin(A+B) = \sin A \cos B + \cos A \sin B$

$\sin(A+B) = \cos[90^\circ - (A+B)]$ — using complementary angles, $\sin\theta = \cos(90^\circ - \theta)$

$= \cos[90^\circ - A - B]$

$= \cos[(90^\circ - A) - B]$

$= \cos(90^\circ - A)\cos B + \sin(90^\circ - A)\sin B$ — using formula 4, $\cos(A-B) = \cos A \cos B + \sin A \sin B$

$= \sin A \cos B + \cos A \sin B$

$\tan(A+B) = \dfrac{\tan A + \tan B}{1 + \tan A \tan B}$

$\tan(A+B) = \dfrac{\sin(A+B)}{\cos(A+B)}$ — $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

$= \dfrac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$

$= \dfrac{\dfrac{\sin A \cancel{\cos B}}{\cos A \cancel{\cos B}} + \dfrac{\cancel{\cos A} \sin B}{\cancel{\cos A} \cos B}}{1 - \dfrac{\sin A \sin B}{\cos A \cos B}}$ — dividing everywhere by $\cos A \cos B$

$= \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

APPLICATION OF FORMULAE 1–24

1. $\cos^2 A + \sin^2 A = 1$	**7.** $\sin(A + B) = \sin A \cos B + \cos A \sin B$
2. Sine Formula: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$	**8.** $\sin(A - B) = \sin A \cos B - \cos A \sin B$
3. Cosine Formula: $a^2 = b^2 + c^2 - 2\,bc \cos A$	**9.** $\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$
4. $\cos(A - B) = \cos A \cos B + \sin A \sin B$	**10.** $\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$
5. $\cos(A + B) = \cos A \cos B - \sin A \sin B$	**11.** $\sin 2A = 2 \sin A \cos A$
6. $\cos 2A = \cos^2 A - \sin^2 A$	**12.** $\sin 2A = \frac{2 \tan A}{1 + \tan^2 A}$

13. $\cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A}$	**19.** $2 \sin A \sin B = \cos(A - B) - \cos(A + B)$
14. $\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$	**20.** $2 \cos A \sin B = \sin(A + B) - \sin(A - B)$
15. $\cos^2 A = \frac{1}{2}(1 + \cos 2A)$	**21.** $\cos A + \cos B = 2 \cos\frac{A + B}{2} \cos\frac{A - B}{2}$
16. $\sin^2 A = \frac{1}{2}(1 - \cos 2A)$	**22.** $\cos A - \cos B = -2 \sin\frac{A + B}{2} \sin\frac{A - B}{2}$
17. $2 \cos A \cos B = \cos(A + B) + \cos(A - B)$	**23.** $\sin A + \sin B = 2 \sin\frac{A + B}{2} \cos\frac{A - B}{2}$
18. $2 \sin A \cos B = \sin(A + B) + \sin(A - B)$	**24.** $\sin A - \sin B = 2 \cos\frac{A + B}{2} \sin\frac{A - B}{2}$