A geometric series has a 'common ratio'.

r= common ratio

eg..

$$4, \xrightarrow{\times 2} 8, \xrightarrow{\times 2} 16, \quad 32, \quad \dots .$$

a= T_1

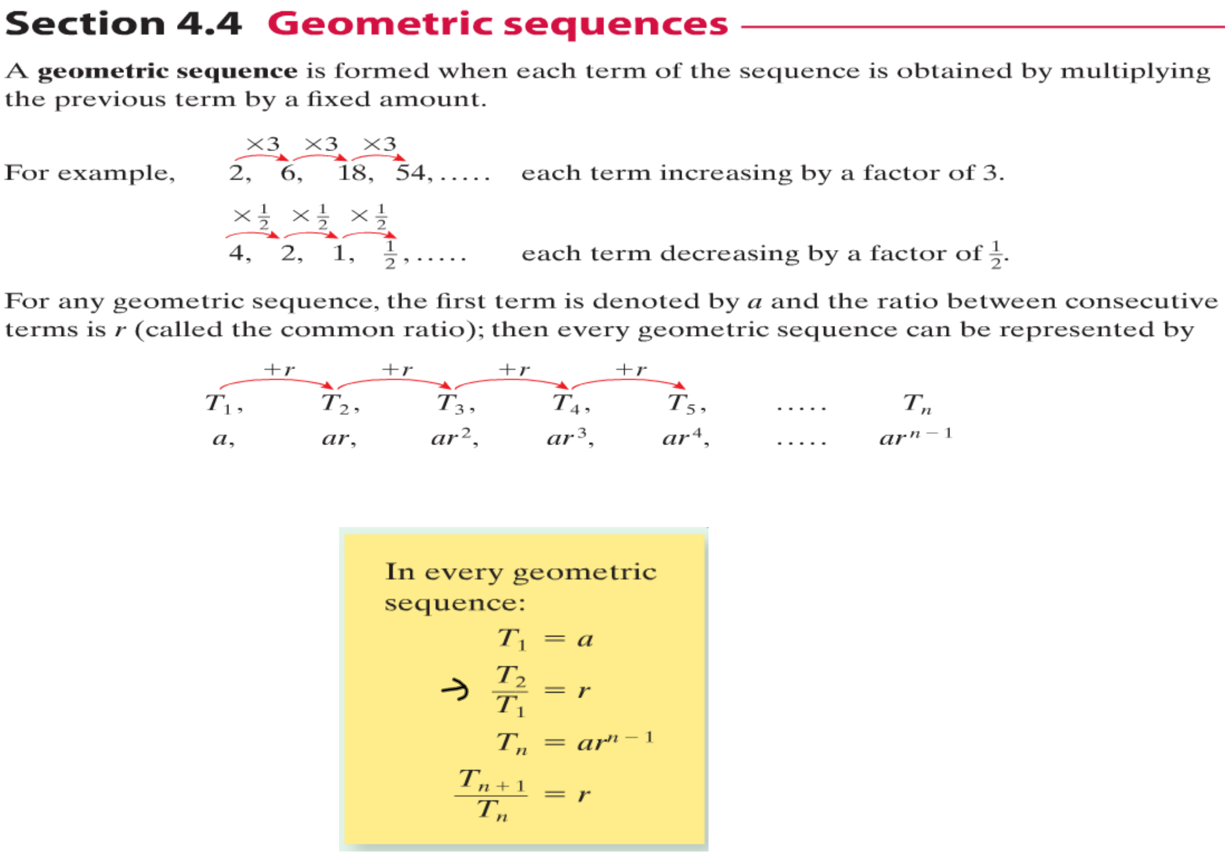
$$a, \quad ar, \quad ar^2, \quad ar^3 \quad \dots$$

Rule: $T_n = ar^{n-1}$

In a geometric series

$$\frac{T_n}{T_{n-1}} = \text{a constant}, \; r$$

r= common ratio



Section 4.4 Geometric sequences

A **geometric sequence** is formed when each term of the sequence is obtained by multiplying the previous term by a fixed amount.

For example, $2 \xrightarrow{\times 3} 6 \xrightarrow{\times 3} 18 \xrightarrow{\times 3} 54, \dots\dots$ each term increasing by a factor of 3.

$4 \xrightarrow{\times \frac{1}{2}} 2 \xrightarrow{\times \frac{1}{2}} 1 \xrightarrow{\times \frac{1}{2}} \frac{1}{2}, \dots\dots$ each term decreasing by a factor of $\frac{1}{2}$.

For any geometric sequence, the first term is denoted by a and the ratio between consecutive terms is r (called the common ratio); then every geometric sequence can be represented by

$$T_1, \xrightarrow{+r} T_2, \xrightarrow{+r} T_3, \xrightarrow{+r} T_4, \xrightarrow{+r} T_5, \quad \dots\dots \quad T_n$$

$$a, \quad ar, \quad ar^2, \quad ar^3, \quad ar^4, \quad \dots\dots \quad ar^{n-1}$$

In every geometric sequence:

$$T_1 = a$$

$$\rightarrow \frac{T_2}{T_1} = r$$

$$T_n = ar^{n-1}$$

$$\frac{T_{n+1}}{T_n} = r$$

Example 1

Find T_n and T_{10} of the geometric sequence $1, \frac{1}{4}, \frac{1}{16}, \frac{1}{64}, \ldots$

$T_n = ar^{n-1}$

$a = 1$ $\quad r = \frac{(\frac{1}{4})}{1} = \frac{1}{4}$

$T_n = 1\left(\frac{1}{4}\right)^{n-1} = \frac{1}{4^{n-1}}$

$T_{10} = \frac{1}{4^9} = \frac{1}{262144}$

Example 2

In a geometric sequence, $\underline{T_3 = 32}$ and $\underline{T_6 = 4}$.

Find a and r and hence write down the first six terms of the sequence.

$T_n = ar^{n-1}$

$ar^2 = 32$ $\quad ar^5 = 4$ ②

$a = \frac{32}{r^2}$ ①

SOLVE

① → ②

$\left(\frac{32}{r^2}\right)r^5 = 4$

$32r^3 = 4$

$r^3 = \frac{4}{32} = \frac{1}{8}$

$r = \sqrt[3]{\frac{1}{8}} = \frac{1}{2}$

→ ① $\quad a = \frac{32}{(\frac{1}{2})^2} = 128$

Example 3

3, x, $x + 6$, ... are the first three terms of a geometric sequence of positive terms. Find

(i) the value of x (ii) the tenth term of the sequence.

$3\ ,\ x\ ,\ x+6$

$a \quad ar \quad ar^2$

Ratio $r = \frac{x}{3} = \frac{x+6}{x}$

$\times 3x$ $x^2 = 3x + 18$

$x^2 - 3x - 18 = 0$

$(x+3)(x-6) = 0$

$x = -3$ or 6

Positive $\Rightarrow$ $x = 6$ $r = \frac{x}{3} = \frac{6}{3} = 2$

Sequence 3, 6, 12 ...

$T_n = ar^{n-1}$ $T_{10} = 3(2)^9 = 1536$

Example 4

The product of the first three terms of a geometric sequence is 216 and their sum is 21. Given that the common ratio r is less than 1, find the first three terms of the sequence.

let terms = $a, ar, ar^2, \ldots$

PRODUCT $(a)(ar)(ar^2) = 216$

$(ar)^3 = 216 \Rightarrow ar = \sqrt[3]{216} = 6$

$\Rightarrow r = 6/a$ ①

sum $a + ar + ar^2 = 21$

$a + a\left(\frac{6}{a}\right) + a\left(\frac{6}{a}\right)^2 = 21$

① $\rightarrow$ $a + 6 + \frac{36}{a} = 21$

$\times a$ $a^2 + 6a + 36 = 21a$

$a^2 - 15a + 36 = 0$

$(a-12)(a-3) = 0$

$a = \textcircled{12}$ or 3 X

$r < 1$ $\Rightarrow r = 6/12 = 1/2$

sequence 12, 6, 3, .

Example 5

Find the number of terms in the geometric sequence $81, 27, 9, \ldots, \frac{1}{27}$ T_n

$n = ?$

$T_n = ar^{n-1}$

$a = 81 \quad r = \frac{1}{3} \quad T_n = \frac{1}{27}$

$\Rightarrow 81\left(\frac{1}{3}\right)^{n-1} = \frac{1}{27}$

$\div 81$

$\left(\frac{1}{3}\right)^{n-1} = \frac{1}{2187}$

$n - 1 = \log_{\boxed{1/3}} \boxed{\frac{1}{2187}} = 7$

$n = 8$

Exponential sequences

Exponential functions of the form $y = Aa^x$, where A is the initial value and a the multiplier or common ratio, produce geometric sequences.

Consider a ball dropping from a height of 10 m.

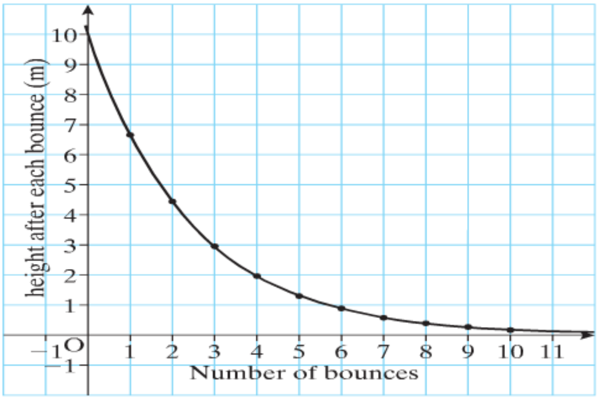
If the ball bounces back to $\frac{2}{3}$ of its original height on each bounce, the height of the ball is given by the following pattern:

After 1 bounce: $10 \times \frac{2}{3} = 10\left(\frac{2}{3}\right)^1$

After 2 bounces: $10 \times \frac{2}{3} \times \frac{2}{3} = 10\left(\frac{2}{3}\right)^2$

After 3 bounces: $10 \times \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} = 10\left(\frac{2}{3}\right)^3$

After n bounces: $10 \times \left(\frac{2}{3}\right)^n$



Example 6

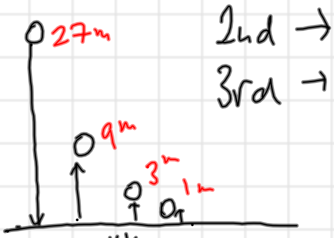
A ball is dropped from a height of 27 m and loses $\frac{2}{3}$ of its height on each bounce.

(i) Find the height of the ball on each of its first four bounces. ✓

(ii) Hence write down the height of the ball after the 10th bounce. ✓

(iii) After which bounce will the ball be at most 2.5 m above the ground?

Picture



2nd → 3 m

3rd → 1 m

$T_n = ar^{n-1}$

Sequence: $\overset{\times 1/3}{27, 9}, 3, 1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27} \ldots$

$a = 27$, $r = \frac{9}{27} = \frac{1}{3}$

$T_{10} = 27\left(\frac{1}{3}\right)^9 = \frac{1}{729}$

Exercise 4.4

1. Determine which of the following sequences are geometric.

Find the common ratios of these sequences and write down the next two terms of each sequence.

(i) 3, 9, 27, 81, … (ii) 1, $\frac{1}{3}$, $\frac{1}{9}$, $\frac{1}{27}$, …

$r = \frac{T_2}{T_1}$

→ To get next term x3

(i) $r = \frac{9}{3} = 3$

$3, 9, 27, 81, \underline{243}, \underline{729}$

(ii) $r = \frac{\left(\frac{1}{3}\right)}{1} = \frac{1}{3}$

$1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \underline{\frac{1}{81}}, \underline{\frac{1}{243}}$

2. Each of the following sequences is geometric.
Find a and r and hence find the indicated term.

(i) 5, 10, … (T_{11}) (ii) 10, 25, … (T_7)

$r = \frac{T_2}{T_1}$

$T_n = ar^{n-1}$

(i) $a = 5$, $r = \frac{10}{5} = 2$

$T_{11} = 5(2)^{10} = 5{,}120$

(ii) $a = 10$, $r = \frac{25}{10} = 2.5$

$T_7 = 10(2.5)^6 = 2441.40625$

3. Given $T_2 = 12$ and $T_5 = 324$, find a and r and hence write down the first five terms of the sequence.

$T_n = ar^{n-1}$

$T_2 = 12 \Rightarrow 12 = ar^1 \Rightarrow a = 12/r$ ①

Solve

$T_5 = 324 \Rightarrow 324 = ar^4$ ②

① → ②

$324 = \left(\frac{12}{\cancel{r}}\right) r^{\cancel{4}3}$

$r^3 = \frac{324}{12} = 27 \Rightarrow r = \sqrt[3]{27} \Rightarrow r = 3$

①

$a = 12/3 = 4$, $a = 4$

First 5 terms: 4, 12, 36, 108, 324

5. Write down the first five terms of the geometric sequence that has a second term 4 and a fifth term $-\frac{1}{16}$.

$T_n = ar^{n-1}$

$T_2 = 4 \Rightarrow 4 = ar^1 \Rightarrow a = 4/r$ ①

$T_5 = -\frac{1}{16} \Rightarrow -\frac{1}{16} = ar^4$ ②

SOLVE

① → ② $\quad -\frac{1}{16} = \left(\frac{4}{\cancel{r}}\right) r^{\cancel{4}3} \Rightarrow -\frac{1}{16} = 4r^3$

÷4 $\quad -\frac{1}{64} = r^3 \Rightarrow r = \sqrt[3]{\left(-\frac{1}{64}\right)} \Rightarrow r = -1/4$

$a = ?$ $\quad T_2 = ar^1 \quad T_2 = 4 \Rightarrow 4 = a\left(-\frac{1}{4}\right) \Rightarrow -16 = a$

First 5 terms $\quad -16, 4, -1, 1/4, -1/16$

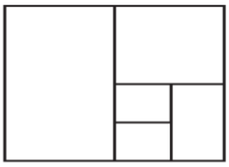
6. A:

......etc.

B:

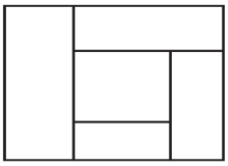
......etc.

C:



......etc.

D:

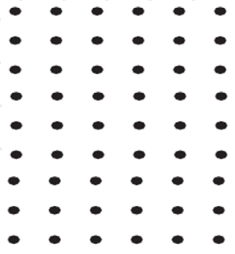


......etc.

By inspection, decide which of the above patterns generate a geometric sequence. Draw the next pattern of those that are geometric.

Only A is geometric $\quad r = 3$

next term ⇒



7. The three numbers $n-2, n$ and $n+3$ are three consecutive terms of a geometric sequence. Find the value of n and hence write down the first four terms of the sequence.

$r = \frac{T_{n+1}}{T_n}$

Solve

$$r = \frac{n}{n-2} = \frac{n+3}{n}$$

$$\Rightarrow \quad n^2 = (n+3)(n-2)$$

$$n^2 = n^2 - 2n + 3n - 6$$

$$0 = n - 6$$

$$n = 6$$

$n = 6$

$a = 4$

$$\Rightarrow \quad n-2 = 6-2=4 \Rightarrow a = 4$$

$$n = 6$$

$$n+3 = 6+3=9$$

$r = \frac{T_{n+1}}{T_n}$

$$r = \frac{\cancel{6}^{3}}{\cancel{4}_{2}} \Rightarrow r = 3/2$$

First 4 terms: $4, 6, 9, 13.5$

8. The third term of a geometric sequence is -63 and the fourth term is 189. Find

(i) the values of a and r

(ii) an expression for T_n.

(i)

$$T_3 = -63$$

$$T_4 = 189$$

$r = \frac{T_4}{T_3}$

$$r = \frac{189}{-63} \Rightarrow r = -3$$

$a = ?$ $\quad T_n = ar^{n-1}$

$$T_3 = -63$$

$$\Rightarrow -63 = a(-3)^2$$

$$-63 = 9a$$

$$a = \frac{-63}{9} \Rightarrow a = -7$$

(ii)

$$T_n = (-7)(-3)^{n-1}$$

9. The first term of a geometric sequence is 16 and the fifth term is 9. What is the value of the seventh term?

$T_1 \xrightarrow{xr} T_2 \xrightarrow{xr} T_3 \xrightarrow{xr} T_4 \xrightarrow{xr} T_5$

$\sqrt{r^4} = \sqrt{\frac{9}{16}}$

$r^2 = \frac{3}{4}$

$r = \sqrt{3/4}$

$T_n = ar^{n \cdot 1}$

$T_1 = a = 16$

$T_5 = 9$

$\Rightarrow 16r^4 = 9$

$r^4 = \frac{9}{16}$; $r = \sqrt[4]{\frac{9}{16}} = \sqrt{\sqrt{\frac{9}{16}}} = \sqrt{\frac{3}{4}}$

$T_7 = 16\left(\sqrt{\frac{3}{4}}\right)^6 = 6.75$ ✓

alternative method

9. The first term of a geometric sequence is 16 and the fifth term is 9. What is the value of the seventh term?

$T_n = ar^{n-1}$

$T_1 = a = 16$

$T_5 = 9$

$9 = 16(r)^4$

$r^4 = \frac{9}{16} \Rightarrow r = \sqrt{\frac{3}{4}}$

$T_n = ar^{n-1}$

$T_7 = 16\left(\sqrt{\frac{3}{4}}\right)^6 = 6.75$

10. The product of the first three terms of a geometric sequence is 27 and their sum is 13. Find the first four terms of the sequence.

let 1st 3 terns be a, ar, ar^2

PRODUCT $a \times ar \times ar^3 = (ar)^3 = 27$

$\Rightarrow ar = \sqrt[3]{27} = 3 \Rightarrow a = \frac{3}{r}$ ①

Sum $a + ar + ar^2 = 13$ ②

Sub in ① → ② $\frac{3}{r} + \left(\frac{3}{r}\right)r + \left(\frac{3}{r}\right)r^2 = 13$

×r $3 + 3r + 3r^2 = 13r$

Solve $3r^2 - 10r + 3 = 0$

$(3r - 1)(r - 3) = 0$

$r = 1/3$ or $r = 3$

Sub into ① $a = \frac{3}{(\frac{1}{3})} = 9$ or $a = \frac{3}{3} = 1$

First TERMS $9, 3, 1, \frac{1}{3}$ or $1, 3, 9, 27$