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Sequences – Series – Patterns

Chapter 4

Key words

number sequence, arithmetic sequence, series, sigma (Σ)
geometric sequence, exponential sequence, geometric series, recurring decimal
finite difference, composite function, quadratic function

6th Year

HL Maths

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Example 1

Write down the first four terms of each of the following sequences:

(i) $T_n = n^2 + n$ (ii) $T_n = 2^n - 3n$

(i) $T_n = n^2 + n$

$T_1 = (1)^2 + (1) = 2$

$T_2 = (2)^2 + 2 = 6$

$T_3 = (3)^2 + 3 = 12$

$T_4 = (4)^2 + 4 = 20$

(ii) $T_n = 2^n - 3n$

$T_1 = 2^1 - 3(1) = -1$

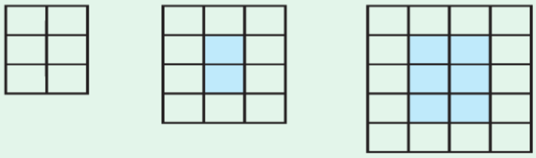
$T_2 = 2^2 - 3(2) = -2$

$T_3 = 2^3 - 3(3) = -1$

$T_4 = 2^4 - 3(4) = 4$

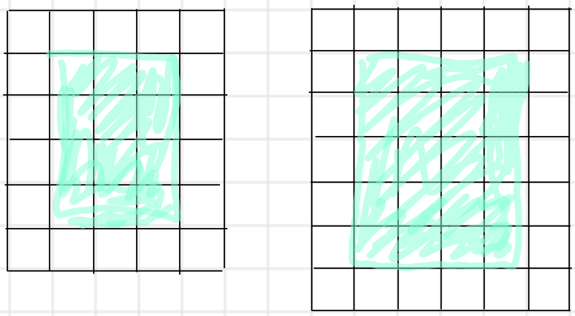
Example 2

The following rectangular patterns are made from two sets of coloured tiles.



(i) Draw the next two patterns of tiles.
(ii) Write a number sequence for the blue tiles used in each of these patterns.
(iii) Write a number sequence for the total number of tiles used in each of these patterns.
(iv) Write a number sequence for the white tiles used in each of these patterns.
(v) Write out the next 3 terms in each sequence found in (ii), (iii), (iv).

(i)



(ii) Blue: 0, 2, 6, 12, 20, 30, 42, 56
(v) (iii) All tiles: 6, 12, 20, 30, 42, 56, 72, 90
(iv) White tiles: 6, 10, 14, 18, 22, 26, 30, 34

Exercise 4.1

1. Write down the next three terms of each of the following sequences:

arithmetic +6 (i) 6, 12, 18, 24, ... 30, 36, 42
(ii) 7, 12, 17, 22, ...
(iii) 4.7, 5.9, 7.1, 8.3, ...
arithmetic −3 (iv) 2, −1, −4, −7, ... −10, −13, −16
(v) 2, 3, 6, 11, 18, 27, ...

arithmetic −8 (vi) 78, 70, 62, 54, ... 46, 38, 30
(vii) 10, 5, 0, −5, −10, ...
(viii) −64, −55, −46, −37, ...
(ix) 2, 6, 18, ...
quadratic +4, +6, +8... (x) 2, 6, 12, 20, ... 30, 42, 56

arithmetic $-\frac{1}{2}$ (xi) $\frac{3}{4}, \frac{1}{4}, -\frac{1}{4}, -\frac{3}{4}, -\frac{5}{4}, -\frac{7}{4}$
(xii) 1, 2, 4, 7, 11, ...
quadratic +3, +5, +7... (xiii) 0, 3, 8, 15, 24, ... 35, 48, 63
geometric x−2 (xiv) 3, −6, 12, −24, ... 48, −96, 192
(xv) $\frac{1}{2}, \frac{1}{6}, \frac{1}{12}, \frac{1}{20}, \ldots$

2. Find the first four terms of the following sequences, given the nth term (T_n) in each case.

(i) $T_n = 4n - 2$ (iv) $T_n = (n + 3)(n + 1)$ (vii) $T_n = 2^n$
(ii) $T_n = (n + 1)^2$ (v) $T_n = n^3 - 1$ (viii) $T_n = (-3)^n$
(iii) $T_n = n^2 - 2n$ (vi) $T_n = \frac{n}{n+2}$ (ix) $T_n = n.2^n$

HW

(i) $T_n = 4n-2$
$T_1 = 4(1)-2 = 2$
$T_2 = 4(2)-2 = 6$
$T_3 = 4(3)-2 = 10$
$T_4 = 4(4)-2 = 14$ ✓

(ii) $T_n = (n+1)^2$
$T_1 = (1+1)^2 = 4$
$T_2 = (2+1)^2 = 9$
$T_3 = (3+1)^2 = 16$
$T_4 = (4+1)^2 = 25$ ✓

(iii) $T_n = n^2-2n$
$T_1 = (1)^2-2(1) = -1$
$T_2 = (2)^2-2(2) = 0$
$T_3 = (3)^2-2(3) = 3$
$T_4 = (4)^2-2(4) = 8$ ✓

(iv) $T_n = (n+3)(n+1)$
$T_1 = (1+3)(1+1) = 8$
$T_2 = (2+3)(2+1) = 15$
$T_3 = (3+3)(3+1) = 24$
$T_4 = (4+3)(4+1) = 35$ ✓

(v) $T_n = n^3-1$
$T_1 = (1)^3-1 = 0$
$T_2 = (2)^3-1 = 7$
$T_3 = (3)^3-1 = 26$
$T_4 = (4)^3-1 = 63$ ✓

(vi) $T_n = n/(n+2)$
$T_1 = 1/(1+2) = 1/3$
$T_2 = 2/(2+2) = 1/2$
$T_3 = 3/(3+2) = 3/5$
$T_4 = 4/(4+2) = 2/3$ ✓

2. Find the first four terms of the following sequences, given the nth term (T_n) in each case.

(i) $T_n = 4n - 2$ (iv) $T_n = (n + 3)(n + 1)$ (vii) $T_n = 2^n$
(ii) $T_n = (n + 1)^2$ (v) $T_n = n^3 - 1$ (viii) $T_n = (-3)^n$
(iii) $T_n = n^2 - 2n$ (vi) $T_n = \frac{n}{n+2}$ (ix) $T_n = n.2^n$

HW

(vii) $T_n = 2^n$
$T_1 = 2^1 = 2$
$T_2 = 2^2 = 4$
$T_3 = 2^3 = 8$
$T_4 = 2^4 = 16$ ✓

(viii) $T_n = (-3)^n$
$T_1 = (-3)^1 = -3$
$T_2 = (-3)^2 = 9$
$T_3 = (-3)^3 = -27$
$T_4 = (-3)^4 = 81$ ✓

(ix) $T_n = n.2^n$
$T_1 = (1)2^1 = 2$
$T_2 = (2)2^2 = 8$
$T_3 = (3)2^3 = 24$
$T_4 = (4)2^4 = 64$ ✓

Arithmetic Sequence

eg.
$T_1 \quad T_2 \quad T_3 \quad T_4 \quad T_5$
$2 \xrightarrow{+2} 4 \xrightarrow{+2} 6 \xrightarrow{+2} 8 \xrightarrow{+2} 10 \xrightarrow{+2} \ldots$

T_n = n^{th} term eg.. $T_3 = 6$
n = n
$a = T_1 = 2$
d = common difference = +2

$T_{20} = ?$ $2 + 19(2)$
$T_{99} = ?$ $2 + 98(2)$
$T_n = ?$ $T_n = 2 + (n-1)2$

Formula $T_n = a + (n-1)d$

Example 1

Find the nth term (T_n) of the arithmetic sequence:

$-2, \quad 3, \quad 8, \quad 13, \ \ldots\ldots$

and hence find (i) T_{20} (ii) T_{21} (iii) $T_{21} - T_{20}$.

$T_n = a + (n-1)d$

$a = -2 \qquad d = 5.$

$n = n$ $T_n = -2 + (n-1)5$
$= -2 + 5n - 5$
$T_n = -7 + 5n$

$n = 20$ (i) $T_{20} = -7 + 5(20) = 93$

(ii) $T_{21} = -7 + 5(21) = 98$

(iii) $T_{21} - T_{20} = 5$

Exercise 4.2

1. Find T_n, the nth term of the following arithmetic sequences. Hence find T_{22} for each sequence.

(i) 8, 13, 18, 23, ... (ii) 16, 36, 56, 76, ... (iii) 10, 7, 4, 1, ...

(i) $a = 8 \quad d = 5$

$T_n = a + (n-1)d$

$T_n = 8 + (n-1)5$

$T_n = 8 + 5n - 5$

$T_n = 3 + 5n$

$n = 22$

$T_{22} = 3 + 5(22) = 113$

(ii) $a = 16 \quad d = 20$

$T_n = 16 + (n-1)20$

$= 16 + 20n - 20$

$T_n = 20n - 4$

$n = 22$

$T_{22} = 20(22) - 4 = 436$

Exercise 4.2

1. Find T_n, the nth term of the following arithmetic sequences. Hence find T_{22} for each sequence.

(i) 8, 13, 18, 23, ... (ii) 16, 36, 56, 76, ... (iii) 10, 7, 4, 1, ...

(iii) 10, 7, 4, 1 ...

$a = 10 \quad d = -3$

$T_n = a + (n-1)d$

$T_n = 10 + (n-1)(-3) = 10 - 3n + 3$

$T_n = 13 - 3n$

$n = 22$

$T_{22} = 13 - 3(22) = -53$ ✓

3. Find the number of terms in each of the following arithmetic sequences:

(i) $-5, -1, 3, 7, \ldots\ldots 75$ (ii) $2, 5, 8, 11, \ldots\ldots 59$ (iii) $-\frac{3}{2}, -1, -\frac{1}{2}, 0, \ldots\ldots 14$.

(i)

$a = -5 \quad d = 4 \quad T_n = 75 \quad n = ?$

$T_n = a + (n-1)d$

$n = ?$

$\Rightarrow 75 = -5 + (n-1)4$

$75 = -5 + 4n - 4$

$75 = -9 + 4n$

$84 = 4n$

$21 = n$ ✓

$\Rightarrow T_{21} = 75$

check: $T_{21} = -5 + (20)4 = 75$ ✓

3. Find the number of terms in each of the following arithmetic sequences:

(i) $-5, -1, 3, 7, \ldots\ldots 75$ (ii) $2, 5, 8, 11, \ldots\ldots 59$ (iii) $-\frac{3}{2}, -1, -\frac{1}{2}, 0, \ldots\ldots 14$.

(ii)

$n = ?$

$a = 2, \quad d = 3, \quad T_n = 59, \quad n = ?$

$T_n = a + (n-1)d$

$59 = 2 + (n-1)3$

-2 $\quad 57 = (n-1)3$

expand $\quad 57 = 3n - 3$

$+3$ $\quad 60 = 3n$

$\div 3$ $\quad 20 = n$

$\Rightarrow T_{20} = 59$ ✓

check: $T_{20} = 2 + (19)3 = 59$ ✓

3. Find the number of terms in each of the following arithmetic sequences:
(i) $-5, -1, 3, 7, \ldots\ldots 75$ (ii) $2, 5, 8, 11, \ldots\ldots 59$ (iii) $-\frac{3}{2}, -1, -\frac{1}{2}, 0, \ldots\ldots 14.$

(iii)

$n = ?$

$T_n = a + (n-1)d$

$a = -\frac{3}{2}$, $d = \frac{1}{2}$, $T_n = 14$, $n = ?$

$\Rightarrow 14 = -\frac{3}{2} + (n-1)(\frac{1}{2})$

$\times 2$ $28 = -3 + (n-1)1$

$+3$ $31 = n - 1$

$+1$ $32 = n$

$\Rightarrow T_{32} = 14$ ✓

check: $T_{32} = -3/2 + (31)(\frac{1}{2}) = 14$ ✓

Example 2

Find the number of terms in the sequence
$1, \quad -3, \quad -7, \quad -11, \quad \ldots\ldots \quad -251.$

4. In an arithmetic sequence, $T_1 = 4$ and $T_7 = 22$. Using simultaneous equations, find
(i) the values of a and d (ii) the first five terms of the sequence (iii) T_{20}.

(i)

$a = ?$
$d = ?$

$T_1 = a = 4$ ✓

$T_7 = 22$

$T_n = a + (n-1)d$

$\Rightarrow 22 = 4 + (7-1)d$

-4: $18 = 6d$

$\div 6$: $3 = d$ ✓

First 5 Terms ? (ii)

Count up in 3's from 4

$4, 7, 10, 13, 16$ ✓

T_{20} ?

simplify rule for T_n

$T_n = 4 + (n-1)3$

$T_{20} = 4 + (19)3 = 61$ ✓

Example 3

In an arithmetic sequence, $T_4 = 6$ and $3T_2 = T_{10}$, find the values of a and d and hence write out the first 6 terms of the sequence.

Example 4

If $p + 2, 2p + 3$ and $5p - 2$ are three consecutive terms of an arithmetic sequence, find the value of $p, p \in R$.

- Given an arithmetic sequence $T_1, T_2, T_3, T_4, T_5, \ldots\ldots T_n$,

  $T_3 - T_2 = T_4 - T_3 = T_5 - T_4 =$ the common difference (d).

  In general terms:

  $T_{n+1} - T_n = d$ (the common difference).

  A corollary to this is as follows:

  To prove that a sequence is arithmetic, we must show that $T_{n+1} - T_n$ is a constant.

- Also, if $T_{n+1} - T_n > 0$, then the sequence is increasing

  if $T_{n+1} - T_n < 0$, then the sequence is decreasing.

  Note, to find T_{n+1}, substitute $(n + 1)$ for n in T_n.

  If $T_n = 3n + 1$,

  $T_{n+1} = 3(n + 1) + 1 = 3n + 4$.

Example 5

Given (i) $T_n = \frac{n+1}{2}$

(ii) $T_n = \frac{2}{n+1}$, determine whether

(a) the sequence is arithmetic or not
(b) the sequence is increasing or decreasing.

Series

GAuss

$1 + 2 + 3 + \ldots\ldots + 100$
$100 + 99 + 98 + \ldots\ldots + 1$
$101 + 101 + \ldots\ldots + 101$

$\frac{(101)(100)}{2} = \frac{10100}{2} = 5050$

$\boxed{S_n = \frac{(\overset{a}{T_1} + \overset{a+(n-1)d}{T_n})(n)}{2}}$

210

Add 1 to 20

$(1+20) = 21$
$\times 20$
$420 \div 2$
210 ✓

6. In an arithmetic sequence, $T_{13} = 27$ and $T_7 = 3T_2$. Find expressions in terms of n for T_{13}, T_7 and T_2 and hence find the values of a and d.
Write down the first six terms of the sequence.

$T_n = a + (n-1)d$

$T_{13} = a + (13-1)d \Rightarrow T_{13} = a + 12d$

$T_7 = a + (7-1)d \Rightarrow T_7 = a + 6d$

$T_2 = a + (2-1)d \Rightarrow T_2 = a + d$

$T_{13} = 27 \Rightarrow$ $a + 12d = 27$ ①

$T_7 = 3T_2 \Rightarrow$ $a + 6d = 3(a + d)$
$a + 6d = 3a + 3d$
$3d = 2a \Rightarrow d = \frac{2}{3}a$ ②

② → ① $a + 12\left(\frac{2}{3}a\right) = 27$
$a + 8a = 27$
$9a = 27 \Rightarrow a = 3$ ③

③ → ② $d = \frac{2}{3}(3) \Rightarrow d = 2$

7. (i) If $2k + 2, 5k - 3$ and $6k$ are three consecutive terms of an arithmetic sequence, find the value of $k, k \in \mathbb{Z}$.
(ii) Given that $4p, -3 - p$ and $5p + 16$ are three consecutive terms of an arithmetic sequence, find the value of $p, p \in \mathbb{Z}$.

(i) $T_n - T_{n-1} = d$

If you subtract subsequent terms you at way get d

$\Rightarrow (5k-3) - (2k+2) = (6k) - (5k-3)$
$5k - 3 - 2k - 2 = 6k - 5k + 3$
$3k - 5 = k + 3$
$2k = 8$
$k = 4$

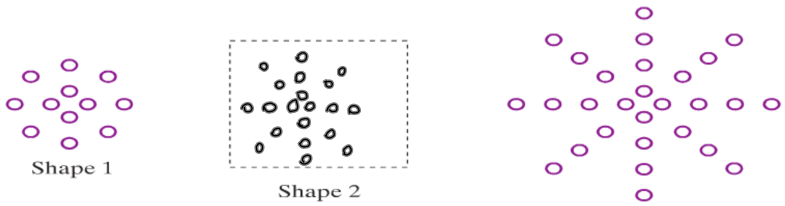
check: $2(4)+2, 5(4)-3, 6(4)$
$10, 17, 24 \ldots$
This is an arithmetic series, $d = 7$

(ii)

$\Rightarrow (-3-p) - (4p) = (5p+16) - (-3-p)$
$-3 - p - 4p = 5p + 16 + 3 + p$
$-3 - 5p = 6p + 19$
$-11p = 22$
$p = -2$

check: $4(-2), -3-(-2), 5(-2)+16$
$-8, -1, 6 \Rightarrow$ series $d = 7$

8.



Shape 1 Shape 2 Shape 3

Three shapes were drawn on a wall.
The second shape was removed accidentally. Given that the shapes were drawn in arithmetic sequence, draw shape 2. ✓

(i) Write a number sequence for the number of circles used in each shape and hence find T_n for the sequence.
(ii) How many circles are needed for shape 15?
(iii) Which shape requires 164 circles?

(i) 12, 20, 28

$T_n = a + (n-1)d$

$T_n = 12 + (n-1)8$
$T_n = 12 + 8n - 8$
$T_n = 4 + 8n$

(ii) $T_{15} = 4 + 8(15) = 124$

$n: ?$ (iii) $164 = 4 + 8n \Rightarrow 160 = 8n$
$n = 20$

The Great Gauss Summation Trick

One of the most famous mathematicians of all times was named Karl Gauss. One day, as the story goes, his teacher gave the class an assignment to keep them busy so that he could take a nap in the back of the class. The problem he assigned would keep most of us busy for at least a half an hour, if not more. However, to his teacher's surprise, young Mr. Gauss solved it in seconds.

Here is the problem the teacher assigned. Students were told to add all the whole numbers from one to one hundred. That is, 1+2+3+4+5 …98+99+100. In less time than it took most students to write out this one hundred number addition problem, Gauss got the answer. The sum is 5,050 he told his teacher confidently, and so it was. But how did he arrive at this answer in so short a time?

$$S_n = \frac{(T_1 + T_n)n}{2}$$

Gauss was a genius, and geniuses sometimes see things differently than most of us non genius types. But that doesn't mean that after being shown the way that we can not solve a problem like a genius would, having first been shown the way. Here is how young Gauss arrived at his answer so quickly. He observed that in the series of numbers 1 +2 + 3 +4 …97 + 98+ 99 + 100, the sum of pairs of numbers from each end, and working in toward the middle summed to the same value,101. In other words, 1 + 100, 2 +99, 3 + 98, 4 + 97 etc. all sum to 101! Since there are fifty pair of numbers in the series 1 to 100, Gauss reasoned that the sum of all the numbers would be 50 times 101 or 5,050.

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$$

Exercise 4.3

1. Find S_n and S_{20} of each of the following arithmetic sequences:

(i) $1 + 5 + 9 + 13 + \ldots$ (ii) $50 + 48 + 46 + 44 + \ldots$
(iii) $1 + 1.1 + 1.2 + 1.3 \ldots$ (iv) $-7 - 3 + 1 + 5 + \ldots$

$S_n = \frac{n}{2}[2a + (n-1)d]$

(i) $a = 1$, $d = 4$

$S_n = \frac{n}{2}[2(1) + (n-1)4] = \frac{n}{2}[2 + 4n - 4]$

$S_n = \frac{n}{2}[4n - 2] = n[2n - 1] = 2n^2 - n$

$S_{20} = 2(20)^2 - (20) = 780$ ✓

$S_n = \frac{n}{2}[2a + (n-1)d]$

(ii) $a = 50$, $d = -2$

$S_n = \frac{n}{2}[2(50) + (n-1)(-2)] = \frac{n}{2}[100 - 2n + 2]$

$S_n = \frac{n}{2}[102 - 2n] = n[51 - n] = 51n - n^2$

$S_{20} = 51(20) - (20)^2 = 620$

Exercise 4.3

1. Find S_n and S_{20} of each of the following arithmetic sequences:

(i) $1 + 5 + 9 + 13 + \ldots$ (ii) $50 + 48 + 46 + 44 + \ldots$
(iii) $1 + 1.1 + 1.2 + 1.3 \ldots$ (iv) $-7 - 3 + 1 + 5 + \ldots$

$S_n = \frac{n}{2}[2a + (n-1)d]$

(iii) $a = 1$, $d = 0.1$

$S_n = \frac{n}{2}[2(1) + (n-1)(0.1)] = \frac{n}{2}[2 + 0.1n - 0.1]$

$S_n = \frac{n}{2}[1.9 + 0.1n] = \frac{19n + n^2}{20}$

$S_{20} = \frac{19(20) + (20)^2}{20} = 19 + 20 = 39$ ✓

$S_n = \frac{n}{2}[2a + (n-1)d]$

(iv) $a = -7$, $d = 4$

$S_n = \frac{n}{2}[2(-7) + (n-1)(4)] = \frac{n}{2}[-14 + 4n - 4]$

$S_n = \frac{n}{2}[4n - 18] = n[2n - 9] = 2n^2 - 9n$

$S_{20} = 2(20)^2 - 9(20) = 620$ ✓

2. Find the sum of each of the following:

(i) $6 + 10 + 14 + 18 + \ldots\ldots 50$ (ii) $1 + 2 + 3 + 4 + \ldots\ldots 100$

(iii) $80 + 74 + 68 + 62 \ldots\ldots -34$

(i)

In these question we first need to discover which term is the last one.

$T_n = a + (n-1)d$

$a = 6, \quad d = 4, \quad T_n = 50$

$n = ?$

$\Rightarrow 50 = 6 + (n-1)4$

$50 = 6 + 4n - 4$

$50 = 2 + 4n$

$48 = 4n$

$12 = n$

$S_n = \frac{n}{2}[2a + (n-1)d]$

$\Rightarrow S_{12} = \frac{12}{2}[2(6) + (12-1)4]$

$= 6[12 + 44] = 6[56]$

$S_{12} = 336$ ✓

2. Find the sum of each of the following:

(i) $6 + 10 + 14 + 18 + \ldots\ldots 50$ (ii) $1 + 2 + 3 + 4 + \ldots\ldots 100$

(iii) $80 + 74 + 68 + 62 \ldots\ldots -34$

In these question we first need to discover which term is the last one.

$T_n = a + (n-1)d$

$a = 1, \quad d = 1, \quad T_n = 100$

$n = ?$

$\Rightarrow 100 = 1 + (n-1)1$

$100 = 1 + n - 1$

$100 = n$

$S_n = \frac{n}{2}[2a + (n-1)d]$

$S_{100} = \frac{100}{2}[2(1) + (100-1)1]$

$= 50[2 + 99] = 50[101]$

$S_{100} = 5050$ ✓

2. Find the sum of each of the following:

(i) $6 + 10 + 14 + 18 + \ldots\ldots 50$ (ii) $1 + 2 + 3 + 4 + \ldots\ldots 100$

(iii) $80 + 74 + 68 + 62 \ldots\ldots -34$

In these question we first need to discover which term is the last one.

$T_n = a + (n-1)d$

$n = ?$

$a = 80, \; d = -6, \; T_n = -34$

$\Rightarrow -34 = 80 + (n-1)(-6)$

$-34 = 80 - 6n + 6$

$-34 = 86 - 6n$

$-120 = -6n$

$20 = n$

$S_n = \frac{n}{2}[2a + (n-1)d]$

$S_{20} = \frac{20}{2}[2(80) + (20-1)(-6)]$

$= 10[160 + 19(-6)]$

$= 10[160 - 114] = 10[46]$

$S_{20} = 460$ ✓

3. How many terms of the series $5 + 8 + 11 + 14 + \ldots$ must be added to make a total of 98?

$S_n = \frac{n}{2}[2a + (n-1)d]$

$a = 5$

$d = 3$

$S_n = 98$

$98 = \frac{n}{2}[2(5) + (n-1)3]$

$2(98) = n[10 + 3n - 3]$

$196 = n[7 + 3n]$

$196 = 7n + 3n^2$

$3n^2 + 7n - 196 = 0$

$(3n + 28)(n - 7) = 0$

$3n + 28 = 0$ | $n = 7$

$n = \frac{-28}{3}$

4. Given $T_n = 5 - 3n$, write down the first term a, and the common difference d. Hence find S_{10}.

$a = 2$

$d = -3$

$T_1 = 5 - 3(1) = 2$

$T_2 = 5 - 3(2) = -1$

$d = T_2 - T_1 = -1 - 2 = -3$

$S_n = \frac{n}{2}[2a + (n-1)d]$

$S_{10} = \frac{10}{2}[2(2) + (10-1)(-3)]$

$= 5[4 - 27]$

$= 5[-23]$

$= -115$

Example 1

Find the sum of the series 4 + 11 + 18 + 25 + ... + 144.

$S_1 = T_1 = 4$

$S_2 = 15$

$T_2 = 15 - 4 = 11$ ✓

$T_2 = S_2 - T_1$

Example 2

To celebrate the birth of his niece, an uncle offers to open a savings account with a deposit of €50. He also offers to every year add €10 more than he did the previous year until his niece is 21 years of age.

(i) Find an expression for S_n, the sum of money on deposit after n years.

(ii) Find S_{21}, the total saved after 21 years.

0 1 21

50 + 60 . . . .

$n=1$ $n=2$ $n=22$

Example 3

Given $S_n = n^2 - 4n$, find an expression for T_n and hence determine if the sequence is arithmetic.

$T_1 = S_1 = (1)^2 - 4(1) = 1 - 4 = -3$

$S_2 = (2)^2 - 4(2) = 4 - 8 = -4$

$a = -3$

$T_2 = S_2 - T_1 = -4 + 3 = -1$

$d = 2$

$-3, -1, 1, 3, \ldots$

$T_n = a + (n-1)d$

$T_n = -3 + (n-1)2$

$= -3 + 2n - 2 = -5 + 2n$

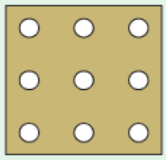
check $T_3 = -5 + 2(3) = -5 + 6 = 1$ ✓

Example 4

A lighting company is making a sequence of light panels with the number of bulbs per panel in arithmetic sequence.

For the first 10 panels, 165 bulbs were used.

If the third panel is as shown in the diagram, find a, the first term of the sequence, and d, the common difference.



3rd panel (9 bulbs)

Hence draw a diagram of the first four panels.

Example 5

(i) Use the sigma notation ($\sum$) to represent $2 + 6 + 10 + 14 + \ldots$ for 45 terms.

(ii) For what value of n is $\sum_{r=1}^{n} 3r - 5 = 90$?

(iii) Find the value of $\sum_{r=1}^{8} 4r - 1$.

$\sum$ = "Sum"

Sigma notation explained.

The Sum of the first 45 terms is usually written as S_{45}.
But this can also be written in "Sigma notation".

$$S_{45} = \sum_{r=1}^{r=45} T_r$$

this means "the sum of terms from term 1 ($r=1$) up to term 45 ($r=45$)"

Example 5

(i) Use the sigma notation ($\sum$) to represent $2 + 6 + 10 + 14 + \ldots\ldots$ for 45 terms.

(ii) For what value of n is $\sum_{r=1}^{n} 3r - 5 = 90$?

(iii) Find the value of $\sum_{r=1}^{8} 4r - 1$.

$T_n = a + (n-1)d$ (i)

$a = 2$, $d = 4$

$S_{45} = \sum_{r=1}^{n} [a + (r-1)d] = \sum_{r=1}^{45} (2 + (r-1)4)$

$= \sum_{r=1}^{45} (2 + 4r - 4) = \sum_{r=1}^{45} 4r - 4$

Example 5

(i) Use the sigma notation ($\sum$) to represent $2 + 6 + 10 + 14 + \ldots\ldots$ for 45 terms.

(ii) For what value of n is $\sum_{r=1}^{n} 3r - 5 = 90$?

(iii) Find the value of $\sum_{r=1}^{8} 4r - 1$.

(ii) $\sum_{r=1}^{n} (3r-5) = 90$, $n = ?$

change to "normal notation"

$T_n = 3n - 5$ $\Rightarrow$ $T_1 = 3(1) - 5 = -2 = a$

$T_2 = 3(2) - 5 = 1$

$d = T_2 - T_1 = 1 - (-2) = 3$ $\Rightarrow$ $d = 3$

$S_n = 90$

$S_n = \frac{n}{2}[2a + (n-1)d]$

$\Rightarrow$ $90 = \frac{n}{2}[2(-2) + (n-1)(3)]$

$180 = n(-4 + 3n - 3)$

$180 = n(-7 + 3n)$

$180 = -7n + 3n^2$

Solve quadratic

$3n^2 - 7n - 180 = 0$

$(3n + 20)(n - 9) = 0$

$n = -20/3$ ✗ or $n = 9$ ✓

Example 5

(i) Use the sigma notation ($\sum$) to represent $2 + 6 + 10 + 14 + \ldots\ldots$ for 45 terms.

(ii) For what value of n is $\sum_{r=1}^{n} 3r - 5 = 90$?

(iii) Find the value of $\sum_{r=1}^{8} 4r - 1$.

(iii)

change to "normal notation"

$T_n = 4n - 1 \quad \Rightarrow T_1 = 4(1) - 1 = 3 = a$

$T_2 = 4(2) - 1 = 7$

$\Rightarrow d = 4$

$n = 8$

$S_n = \frac{n}{2}[2a + (n-1)d]$

$S_8 = \frac{8}{2}[2(3) + (8-1)(4)]$

$= 4[6 + 28] = 4[34]$

$= 136$

5. Anna saves money each week to buy a printer which costs €190. Her plan is to start with €10 and to put aside €2 more each week (i.e. €12, €14, etc.) until she has enough money to buy the printer.
At this rate, how many weeks will it take Anna to save for the printer?

Series $= 10 + 12 + 14 + \ldots$

$a = 10, \quad d = 2, \quad S_n = 190, \quad n = ?$

$S_n = \frac{n}{2}[2a + (n-1)d]$

$190 = \frac{n}{\cancel{2}}[\cancel{2}(10) + (n-1)\cancel{2}]$

$190 = n[10 + n - 1]$

$190 = n[9 + n]$

$190 = 9n + n^2$

Solve quadratic

$\Rightarrow n^2 + 9n - 190 = 0$

$(n + 19)(n - 10) = 0$

$\Rightarrow n = -19$ ✗ or $n = 10$ ✓

−19 weeks makes no sense!

6. Evaluate (i) $\sum_{r=1}^{6}(3r+1)$ (ii) $\sum_{r=0}^{5}(4r-1)$ (iii) $\sum_{r=1}^{100} r$

(i)

$S_6 = ?$

$T_1 = 3(1)+1 = 4 = a$

$T_2 = 3(2)+1 = 7$

$\Rightarrow d = 3$

$n = 6$

$S_n = \frac{n}{2}[2a+(n-1)d]$

$S_6 = \frac{6}{2}[2(4)+(6-1)3]$

$= 3[8+(5)3] = 3[8+15]$

$= 3[23] = 69$ ✓

6. Evaluate (i) $\sum_{r=1}^{6}(3r+1)$ (ii) $\sum_{r=0}^{5}(4r-1)$ (iii) $\sum_{r=1}^{100} r$

(ii)

$\sum_{r=0}^{5}(4r-1)$

Careful here:
In this example the first term is when $r=0$!
We add terms 'till $r=5$
$\Rightarrow$ that means 6 terms.

$T_1 = 4(0)-1 = -1$

$T_2 = 4(1)-1 = 3$

$\Rightarrow d = 4$

Sum of first 6 terms?

$S_n = \frac{n}{2}[2a+(n-1)d]$

$S_6 = \frac{6}{2}[2(-1)+(6-1)4]$

$= 3[-2+(5)4] = 3[-2+20]$

$= 3[18] = 54$ ✓

6. Evaluate (i) $\sum_{r=1}^{6}(3r+1)$ (ii) $\sum_{r=0}^{5}(4r-1)$ (iii) $\sum_{r=1}^{100} r$

$T_1 = 1 = a$
$T_2 = 2$
$\Rightarrow$ $d = 1$
$n = 100$

$S_n = \frac{n}{2}[2a+(n-1)d]$

$S_{100} = \frac{100}{2}[2(1) + (100-1)1]$

$= 50[2+99] = 50[101]$

$= 5050$ ✓

7. Write each of the following series in sigma notation.
(i) $4 + 8 + 12 + 16 + \ldots\ldots 124$ (ii) $-10 - 9\frac{1}{2} - 8 - 7\frac{1}{2} + \ldots\ldots 4$
(iii) $10 + 10.1 + 10.2 + 10.3 + \ldots\ldots 50$

(i) $a = 4$, $d = 4$, $T_n = 124$, $n = ?$

expression for T_n

$T_n = a + (n-1)d$

$T_n = 4 + (n-1)4$
$= 4 + 4n - 4$
$T_n = 4n$

$n = ?$

$124 = 4n$
$n = 31$

series in sigma notation

$\sum_{r=1}^{31} 4r$ ✓

7. Write each of the following series in sigma notation.

(i) $4 + 8 + 12 + 16 + \ldots\ldots 124$ (ii) $-10 - 9\frac{1}{2} - 8 - 7\frac{1}{2} + \ldots\ldots 4$

(iii) $10 + 10.1 + 10.2 + 10.3 + \ldots\ldots 50$

(ii)

$a = -10$, $d = \frac{1}{2}$, $T_n = 4$, $n = ?$

expression for T_n

$T_n = a + (n-1)d$

$T_n = -10 + (n-1)\frac{1}{2}$

$T_n = -10 + \frac{n}{2} - \frac{1}{2}$

$T_n = -10\frac{1}{2} + \frac{n}{2} \Rightarrow T_n = \frac{n-21}{2}$

$T_n = 4 = \frac{n-21}{2}$

$n = ?$

$\Rightarrow 8 = n - 21$

$29 = n$

series in sigma notation

$$\sum_{r=1}^{29} \frac{r-21}{2}$$ ✓

7. Write each of the following series in sigma notation.

(i) $4 + 8 + 12 + 16 + \ldots\ldots 124$ (ii) $-10 - 9\frac{1}{2} - 8 - 7\frac{1}{2} + \ldots\ldots 4$

(iii) $10 + 10.1 + 10.2 + 10.3 + \ldots\ldots 50$

(iii)

$a = 10$, $d = 0.1$, $T_n = 50$

expression for T_n

$T_n = a + (n-1)d$

$T_n = 10 + (n-1)0.1$

$T_n = 10 + 0.1n - 0.1$

$T_n = 9.9 + 0.1n$

$T_n = \frac{99+n}{10}$

$n = ?$

$T_n = \frac{99+n}{10} = 50$

$\Rightarrow 99 + n = 500$

$n = 401$

series in sigma notation

$$\sum_{n=1}^{401} \frac{99+r}{10}$$ ✓

note: mistake in book answer

8. In an arithmetic series, $T_4 = 15$ and $S_5 = 55$.
Find the first five terms of the series.

$T_n = a + (n-1)d$

$T_4 = a + (4-1)d = 15$

$a + 4d - 4 = 15$

$a + 4d = 19$ (1)

$S_n = \frac{n}{2}[2a + (n-1)d]$

$S_5 = \frac{5}{2}[2a + (5-1)d] = 55$

$\Rightarrow 5[2a + 4d] = 110$

$\Rightarrow 2a + 4d = 22$

$a + 2d = 11$ (2)

Solve

(1) − (2) $\Rightarrow 2d = 8 \Rightarrow d = 4$ (3)

(3) → (2) $\Rightarrow a + 2(4) = 11$

$a = 11 - 8 \Rightarrow a = 3$

$\Rightarrow$ First 5 terms $3 + 7 + 11 + 15 + 19 \ldots$

9. The third term of an arithmetic sequence is 18 and the seventh term is 30.
Find the sum of the first 33 terms.

$T_n = a + (n-1)d$

$T_3 = a + (3-1)d = 18$

$a + 2d = 18$ (1)

$T_7 = a + (7-1)d = 30$

$a + 6d = 30$ (2)

Solve

(1) − (2) $\Rightarrow 4d = 12 \Rightarrow d = 3$ (3)

(3) → (2) $\Rightarrow a + 6(3) = 30$

$a = 30 - 18 \Rightarrow a = 12$

$S_n = \frac{n}{2}[2a + (n-1)d]$

$S_{33} = \frac{33}{2}[2(12) + (33-1)3]$

$= \frac{33}{2}[24 + 96] = \frac{33}{2}[120]$

$= 1980$ ✓

12. Show that the sum of the natural numbers from 1 to n is $\frac{n}{2}(n+1)$ and use the formula to find the sum of $1 + 2 + 3 + 4 + \ldots\ldots 99$.

$$1 + 2 + 3 + \ldots + n$$

$a = 1$, $d = 1$, $n = n$

$S_n = \frac{n}{2}[2a + (n-1)d]$

$$S_n = \frac{n}{2}[2(1) + (n-1)1]$$

$$S_n = \frac{n}{2}[2 + n - 1]$$

$$S_n = \frac{n}{2}(n+1)$$

$$S_{99} = \frac{99}{2}(99+1) = 4950$$

14. In an arithmetic sequence, $T_{21} = 37$ and $S_{20} = 320$. Find the sum of the first ten terms.

$T_n = a + (n-1)d$

$$T_{21} = 37$$

$$\Rightarrow 37 = a + (21-1)d$$

$$a + 20d = 37 \quad ①$$

$S_n = \frac{n}{2}[2a + (n-1)d]$

$$S_{20} = 320$$

$$\Rightarrow 320 = \frac{20}{2}[2a + (20-1)d]$$

$$32 = 2a + 19d \quad ②$$

2① − ②

$$\Rightarrow \cancel{2a} + 40d = 74$$

$$-\cancel{2a} - 19d = 32$$

$$21d = 42 \Rightarrow d = 2 \quad ③$$

③ → ①

$$\Rightarrow a + 20(2) = 37 \Rightarrow a = -3$$

$S_n = \frac{n}{2}[2a + (n-1)d]$

$$S_{10} = \frac{10}{2}[2(-3) + (10-1)2]$$

$$= 5[-6 + 18] = 5[12] = 60$$

15. Show that $S_n = \frac{n(a+l)}{2}$ is the sum to n terms of an arithmetic sequence where l is the last term.

$l = \quad T_n = a + (n-1)d$

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$$
$$= \frac{n}{2}\left[a + (a + (n-1)d)\right]$$
$$= \frac{n}{2}\left[a + l\right] \quad ✓$$

16. Explain why S_∞ (the sum to infinity) for an arithmetic sequence cannot be found.

$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$

$\Rightarrow$ "$S_\infty = \frac{\infty}{2}\left[2a + (\infty - 1)d\right]$"

this is problematic because
$\infty/2$ makes no sense
as does $(\infty - 1)$
as does $(\infty - 1)d$
etc....

$\Rightarrow$ to sum you need an end term and this is not the case in S_∞

r = common ratio

$a = T_1$

A geometric series has a 'common ratio'.

eg..

$$4 \xrightarrow{\times 2} 8 \xrightarrow{\times 2} 16, \; 32, \; \ldots$$

$$a, \; ar, \; ar^2, \; ar^3 \; \ldots$$

Rule:

$$T_n = ar^{n-1}$$

In a geometric series

r = common ratio

$$\frac{T_n}{T_{n-1}} = \text{a constant}, \; r$$

Section 4.4 Geometric sequences

A **geometric sequence** is formed when each term of the sequence is obtained by multiplying the previous term by a fixed amount.

For example, $2 \xrightarrow{\times 3} 6 \xrightarrow{\times 3} 18 \xrightarrow{\times 3} 54, \ldots\ldots$ each term increasing by a factor of 3.

$4 \xrightarrow{\times \frac{1}{2}} 2 \xrightarrow{\times \frac{1}{2}} 1 \xrightarrow{\times \frac{1}{2}} \frac{1}{2}, \ldots\ldots$ each term decreasing by a factor of $\frac{1}{2}$.

For any geometric sequence, the first term is denoted by a and the ratio between consecutive terms is r (called the common ratio); then every geometric sequence can be represented by

$$\begin{array}{ccccccc} T_1, & \xrightarrow{\times r} T_2, & \xrightarrow{\times r} T_3, & \xrightarrow{\times r} T_4, & \xrightarrow{\times r} T_5, & \ldots\ldots & T_n \\ a, & ar, & ar^2, & ar^3, & ar^4, & \ldots\ldots & ar^{n-1} \end{array}$$

In every geometric sequence:

$$T_1 = a$$

$$\rightarrow \frac{T_2}{T_1} = r$$

$$T_n = ar^{n-1}$$

$$\frac{T_{n+1}}{T_n} = r$$

Example 1

Find T_n and T_{10} of the geometric sequence $1, \frac{1}{4}, \frac{1}{16}, \frac{1}{64}, \ldots$

$T_n = ar^{n-1}$

$a = 1 \qquad r = \frac{\left(\frac{1}{4}\right)}{1} = \frac{1}{4}$

$T_n = 1\left(\frac{1}{4}\right)^{n-1} = \frac{1}{4^{n-1}}$

$T_{10} = \frac{1}{4^9} = \frac{1}{262144}$

Example 2

In a geometric sequence, $T_3 = 32$ and $T_6 = 4$.

Find a and r and hence write down the first six terms of the sequence.

$T_n = ar^{n-1}$

$ar^2 = 32 \qquad ar^5 = 4 \quad (2)$

$a = \frac{32}{r^2} \quad (1)$

SOLVE

(1) → (2)

$\left(\frac{32}{r^2}\right) r^5 = 4$

$32r^3 = 4$

$r^3 = \frac{4}{32} = \frac{1}{8}$

$r = \sqrt[3]{\frac{1}{8}} = \frac{1}{2}$

→ (1) $a = \frac{32}{\left(\frac{1}{2}\right)^2} = 128$

Example 3

$3, x, x+6, \ldots$ are the first three terms of a geometric sequence of positive terms.
Find
(i) the value of x (ii) the tenth term of the sequence.

$3, \quad x, \quad x+6$

$a, \quad ar, \quad ar^2$

Ratio $\quad r = \frac{x}{3} = \frac{x+6}{x}$

$\times 3x \quad x^2 = 3x + 18$

$x^2 - 3x - 18 = 0$

$(x+3)(x-6) = 0$

$x = -3$ or 6

Positive $\Rightarrow$ $x = 6$ $\quad r = \frac{x}{3} = \frac{6}{3} = 2$

Sequence $\quad 3, 6, 12 \ldots$

$T_n = ar^{n-1}$ $\quad T_{10} = 3(2)^9 = 1536$

Example 4

The product of the first three terms of a geometric sequence is 216 and their sum is 21. Given that the common ratio r is less than 1, find the first three terms of the sequence.

let terms = $a, ar, ar^2, \ldots$

PRODUCT $\quad (a)(ar)(ar^2) = 216$

$(ar)^3 = 216 \Rightarrow ar = \sqrt[3]{216} = 6$

$\Rightarrow r = 6/a \quad (1)$

sum $\quad a + ar + ar^2 = 21$

$a + a\left(\frac{6}{a}\right) + a\left(\frac{6}{a}\right)^2 = 21$

$(1) \rightarrow \quad a + 6 + \frac{36}{a} = 21$

$\times a \quad a^2 + 6a + 36 = 21a$

$a^2 - 15a + 36 = 0$

$(a-12)(a-3) = 0$

$a = 12$ or $\cancel{3}$

$r < 1 \quad \Rightarrow r = 6/12 = 1/2$

sequence $\quad 12, 6, 3, .$

Example 5

Find the number of terms in the geometric sequence $81, 27, 9, \ldots, \frac{1}{27}$ T_n

$n = ?$

$T_n = ar^{n-1}$

$a = 81 \quad r = \frac{1}{3} \quad T_n = \frac{1}{27}$

$\Rightarrow 81\left(\frac{1}{3}\right)^{n-1} = \frac{1}{27}$

$\div 81$

$\left(\frac{1}{3}\right)^{n-1} = \frac{1}{2187}$

$n - 1 = \log_{\boxed{1/3}} \boxed{\frac{1}{2187}} = 7$

$n = 8$

Exponential sequences

Exponential functions of the form $y = Aa^x$, where A is the initial value and a the multiplier or common ratio, produce geometric sequences.

Consider a ball dropping from a height of 10 m.

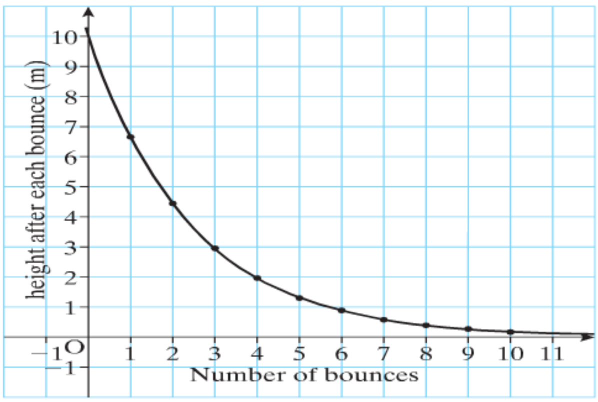
If the ball bounces back to $\frac{2}{3}$ of its original height on each bounce, the height of the ball is given by the following pattern:

After 1 bounce: $10 \times \frac{2}{3} = 10\left(\frac{2}{3}\right)^1$

After 2 bounces: $10 \times \frac{2}{3} \times \frac{2}{3} = 10\left(\frac{2}{3}\right)^2$

After 3 bounces: $10 \times \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} = 10\left(\frac{2}{3}\right)^3$

After n bounces: $10 \times \left(\frac{2}{3}\right)^n$



Example 6

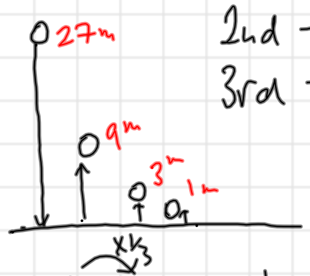
A ball is dropped from a height of 27 m and loses $\frac{2}{3}$ of its height on each bounce.

(i) Find the height of the ball on each of its first four bounces.

(ii) Hence write down the height of the ball after the 10^{th} bounce.

(iii) After which bounce will the ball be at most 2.5 m above the ground?

Picture



2nd → 3 m

3rd → 1 m

Sequence: $27, 9, 3, 1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27} \ldots$

$a = 27, \quad r = \frac{9}{27} = \frac{1}{3}$

$T_n = ar^{n-1}$

$T_{10} = 27\left(\frac{1}{3}\right)^9 = \frac{1}{729}$

Exercise 4.4

1. Determine which of the following sequences are geometric.

Find the common ratios of these sequences and write down the next two terms of each sequence.

(i) 3, 9, 27, 81, ... (ii) $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$

$r = \frac{T_2}{T_1}$

→ To get next term x3

(i) $r = \frac{9}{3} = 3$

$3, 9, 27, 81, \underline{243}, \underline{729}$

(ii) $r = \frac{\left(\frac{1}{3}\right)}{1} = \frac{1}{3}$

$1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \underline{\frac{1}{81}}, \underline{\frac{1}{243}}$

2. Each of the following sequences is geometric.
Find a and r and hence find the indicated term.

(i) 5, 10, … (T_{11}) (ii) 10, 25, … (T_7)

$r = \frac{T_2}{T_1}$

$T_n = ar^{n-1}$

(i) $a = 5$, $r = \frac{10}{5} = 2$

$T_{11} = 5(2)^{10} = 5{,}120$

(ii) $a = 10$, $r = \frac{25}{10} = 2.5$

$T_7 = 10(2.5)^6 = 2441.40625$

3. Given $T_2 = 12$ and $T_5 = 324$, find a and r and hence write down the first five terms of the sequence.

$T_n = ar^{n-1}$

$T_2 = 12 \Rightarrow 12 = ar^1 \Rightarrow a = 12/r$ (1)

Solve

$T_5 = 324 \Rightarrow 324 = ar^4$ (2)

(1) → (2)

$324 = \left(\frac{12}{\cancel{r}}\right) r^{\cancel{4}3}$

$r^3 = \frac{324}{12} = 27 \Rightarrow r = \sqrt[3]{27} \Rightarrow r = 3$

(1) $a = 12/3 = 4$, $a = 4$

First 5 terms: 4, 12, 36, 108, 324

5. Write down the first five terms of the geometric sequence that has a second term 4 and a fifth term $-\frac{1}{16}$.

$T_n = ar^{n-1}$

$T_2 = 4 \Rightarrow 4 = ar^1 \Rightarrow a = 4/r$ ①

$T_5 = -\frac{1}{16} \Rightarrow -\frac{1}{16} = ar^4$ ②

SOLVE

① → ②

$-\frac{1}{16} = \left(\frac{4}{\cancel{r}}\right) r^{\cancel{4}3} \Rightarrow -\frac{1}{16} = 4r^3$

÷4

$-\frac{1}{64} = r^3 \Rightarrow r = \sqrt[3]{\left(-\frac{1}{64}\right)} \Rightarrow r = -1/4$

$a = ?$ $T_2 = ar^1$

$T_2 = 4 \Rightarrow 4 = a\left(-\frac{1}{4}\right) \Rightarrow -16 = a$

First 5 terms

$-16, 4, -1, 1/4, -1/16$

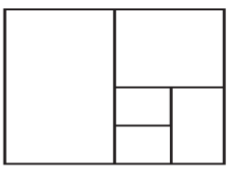
6. A:

......etc.

B:

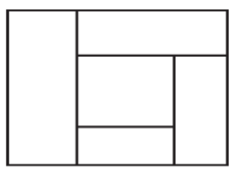
......etc.

C:



......etc.

D:

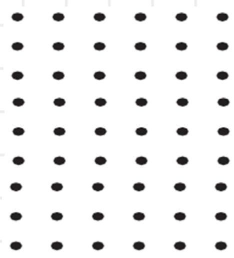


......etc.

By inspection, decide which of the above patterns generate a geometric sequence. Draw the next pattern of those that are geometric.

Only A is geometric $r = 3$

next term =



7. The three numbers $n-2, n$ and $n+3$ are three consecutive terms of a geometric sequence. Find the value of n and hence write down the first four terms of the sequence.

$r = \frac{T_{n+1}}{T_n}$

Solve

$$r = \frac{n}{n-2} = \frac{n+3}{n}$$

$$\Rightarrow \quad n^2 = (n+3)(n-2)$$

$$n^2 = n^2 - 2n + 3n - 6$$

$$0 = n - 6$$

$$n = 6$$

$n=6$

$a=4$

$\Rightarrow \quad n-2 = 6-2 = 4 \Rightarrow a = 4$

$n = 6$

$n+3 = 6+3 = 9$

$r = \frac{T_{n+1}}{T_n}$

$r = \frac{\cancel{6}\,3}{\cancel{4}\,2} \Rightarrow r = 3/2$

First 4 terms: $4, 6, 9, 13.5$

8. The third term of a geometric sequence is -63 and the fourth term is 189. Find

(i) the values of a and r

(ii) an expression for T_n.

(i)

$T_3 = -63$

$T_4 = 189$

$r = \frac{T_4}{T_3}$

$r = \frac{189}{-63} \Rightarrow r = -3$

$a = ?$ $\quad T_n = ar^{n-1}$

$T_3 = -63$

$\Rightarrow -63 = a(-3)^2$

$-63 = 9a$

$a = \frac{-63}{9} \Rightarrow a = -7$

(ii)

$T_n = (-7)(-3)^{n-1}$

9. The first term of a geometric sequence is 16 and the fifth term is 9. What is the value of the seventh term?

$\xrightarrow{\times r}$ $\xrightarrow{\times r}$ $\xrightarrow{\times r}$ $\xrightarrow{\times r}$

T_1, T_2, T_3, T_4, T_5

$T_1 = a = 16$

$T_5 = 9$

$\Rightarrow 16r^4 = 9$

$r^4 = \frac{9}{16}$; $r = \sqrt[4]{\frac{9}{16}} = \sqrt{\sqrt{\frac{9}{16}}} = \sqrt{\frac{3}{4}}$

$T_7 = 16\left(\sqrt{\frac{3}{4}}\right)^6 = 6.75$ ✓

$\sqrt{r^4} = \sqrt{\frac{9}{16}}$

$r^2 = \frac{3}{4}$

$r = \sqrt{3/4}$

$T_n = ar^{n-1}$

alternative method

9. The first term of a geometric sequence is 16 and the fifth term is 9. What is the value of the seventh term?

$T_n = ar^{n-1}$

$T_1 = a = 16$

$T_5 = 9$

$9 = 16(r)^4$

$r^4 = \frac{9}{16} \Rightarrow r = \sqrt{\frac{3}{4}}$

$T_n = ar^{n-1}$

$T_7 = 16\left(\sqrt{\frac{3}{4}}\right)^6 = 6.75$

10. The product of the first three terms of a geometric sequence is 27 and their sum is 13. Find the first four terms of the sequence.

let 1st 3 terns be $a,\ ar,\ ar^2$

Product $a \times ar \times ar^3 = (ar)^3 = 27$

$\Rightarrow ar = \sqrt[3]{27} = 3 \Rightarrow a = \frac{3}{r}$ (1)

Sum $a + ar + ar^2 = 13$ (2)

Sub in (1) → (2) $\frac{3}{r} + \left(\frac{3}{r}\right)r + \left(\frac{3}{r}\right)r^2 = 13$

×r $3 + 3r + 3r^2 = 13r$

Solve $3r^2 - 10r + 3 = 0$

$(3r - 1)(r - 3) = 0$

$r = 1/3$ or $r = 3$

Sub into (1) $a = \frac{3}{\left(\frac{1}{3}\right)} = 9$ or $a = \frac{3}{3} = 1$

First TERMS $9, 3, 1, \frac{1}{3}$ or $1, 3, 9, 27$

Sum to n terms of geometric series

eg.(i) T_5 and S_5 = ?

(i) $1 + 3 + 9 + \ldots$ $a = 1,\ r = 3$

$T_n = ar^{n-1}$

$T_5 = (1)(3)^{5-1} = 3^4 = 81$ ✓

$S_n = \frac{a(1-r^n)}{1-r}$

$S_5 = \frac{1(1-3^5)}{1-3} = \frac{1-3^5}{-2} = \frac{1-243}{-2} = 121$ ✓

Geometric series

$r = \frac{T_2}{T_1}$

$T_n = ar^{n-1}$

$S_n = \frac{a(1-r^n)}{1-r}$

Example 1

Find T_5 and S_5 of each of the following:

(i) $1 + 3 + 9 + \ldots$

$a = 1 \qquad r = 3/1 = 3$

$T_5 = 1(3)^4 = 81$

$S_5 = \frac{1(1-3^5)}{1-3}$

$= 121$

ii) $1 + \frac{1}{4} + \frac{1}{16} + \ldots$

$a = 1 \qquad r = \frac{(1/4)}{1} = 1/4$

$T_5 = 1\left(\frac{1}{4}\right)^4 = \frac{1}{256}$

$S_5 = \frac{1\left(1-\left(\frac{1}{4}\right)^5\right)}{1-\left(\frac{1}{4}\right)}$

$= \frac{341}{256}$

Geometric series

$T_n = ar^{n-1}$

divide

$S_n = \frac{a(1-r^n)}{1-r}$

Example 2

In a geometric series, $T_3 = 32$ and $T_6 = 4$; find a and r and hence find S_8, the sum of the first eight terms.

$T_3 = 32 \qquad T_6 = 4$

$32 = ar^2 \qquad 4 = ar^5$

$\frac{\cancel{a}r^5}{\cancel{a}r^2} = \frac{4}{32} \Rightarrow r^3 = \frac{1}{8}$

$\Rightarrow \sqrt[3]{\frac{1}{8}} = r = \frac{1}{2}$

$\Rightarrow a = 32/\left(\frac{1}{2}\right)^2 = 32(4) = 128 = a$

$S_8 = \frac{128\left(1-\left(\frac{1}{2}\right)^8\right)}{1-\frac{1}{2}} = 255$

P.160

Ex. 4.5

Q4 Series: $32 + 16 + 8 + \dots$ $S_{10} = ?$

$a = 32$ $r = \frac{16}{32} = \frac{1}{2}$

$S_n = \frac{a(1-r^n)}{1-r}$

$S_{10} = \frac{32\left(1 - \left(\frac{1}{2}\right)^{10}\right)}{1 - \frac{1}{2}}$

$\simeq 63.94$

Q5 Series: $4 - 12 + 36 - 108 + \dots$

$S_6 = ?$

$a = 4$ $r = \frac{-12}{4} = -3$

$S_n = \frac{a(1-r^n)}{1-r}$

$S_6 = \frac{4\left(1 - (-3)^6\right)}{1 - (-3)}$

$= -728$

Q6

Series: $729 - 243 + 81 - \dots - \frac{1}{3}$

Find the number of terms?

$n = ?$

$a = 729 \quad , \quad r = \frac{-243}{729} = -\frac{1}{3} \quad , \quad T_n = -\frac{1}{3}$

$T_n = ar^{n-1}$

$\Rightarrow -\frac{1}{3} = 729\left(-\frac{1}{3}\right)^{n-1}$

divide by 729

$\frac{1}{(-3)(729)} = \frac{1}{(-3)^{n-1}}$

$(3)(729) = 2187$

$\log_3 2187 = 7$

$\Rightarrow \frac{1}{(-3)^7} = \frac{1}{(-3)^{n-1}} \quad \Rightarrow n-1 = 7$

$\Rightarrow n = 8$

$S_n = \frac{a(1-r^n)}{1-r}$

$S_8 = \frac{729\left(1-\left(-\frac{1}{3}\right)^8\right)}{1-\left(-\frac{1}{3}\right)} = 546\frac{2}{3}$

HW ex 4.5 Q4-7

Q7

$\sum_{r=1}^{6} 4^r$

Sum terms from input r=1 to r=6

T_n

$T_1 = 4^1 = 4$

$T_2 = 4^2 = 16$ $\times 4$

$T_3 = 4^3 = 64$

$a = 4$

$r = 4$

$S_n = \frac{a(1-r^n)}{1-r}$

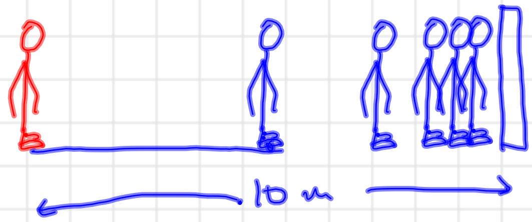
$S_6 = \frac{4(1-4^6)}{1-4} = 5460$

For a geometric series with $|r| < 1$,
$\lim_{n \to \infty} S_n = \frac{a}{1-r}$.

$|r| < 1$

The idea of a sum of infinite terms having a limit.

If I walk towards a wall that is 10 m away and every second I cover half the distance between me and the wall. I will never reach the wall. The sum of all the distances I cover will add up to slightly less than 10 m!



For a geometric series with $|r| < 1$,
$\lim_{n \to \infty} S_n = \frac{a}{1-r}$.

Example 3

Find the sum to infinity of the geometric series $16 + 12 + 9 + \ldots$..

$a = 16$ $\quad r = \frac{12}{16} = \frac{3}{4}$

$$S_\infty = \frac{a}{1-r} = \frac{16}{1-\frac{3}{4}} = \frac{16}{\frac{1}{4}} = 64$$

19. The value of a sum of money on deposit at 3% per annum compound interest is given by $A = €4000\,(1.03)^t$ where t is the number of years of the investment. Find

(i) the amount of money on deposit
(ii) the value of the investment at the end of each of the first four years
(iii) the value of the investment at the end of the 10th year
(iv) the number of years, correct to the nearest year, needed for the investment to double in value.

(i) on deposit $\Rightarrow t = 0$ years

$\Rightarrow A_0 = 4000\,(1.03)^0 = €\,4000$

(ii) 1 year $\Rightarrow A_1 = 4000(1.03)^1 = €\,4120$
2 year $\Rightarrow A_2 = 4000\,(1.03)^2 = €\,4243.60$
3 year $\Rightarrow A_3 = 4000\,(1.03)^3 = €\,4370.91$
4 years $\Rightarrow A_4 = 4000\,(1.03)^4 = €\,4502.04$

(iii) $A_{10} = 4000\,(1.03)^{10} = €\,5375.67$

$A_0 = €4000$
Double $A_0 = €8000$
$t = ?$

$A = 4000\,(1.03)^t$
$\Rightarrow 8000 = 4000\,(1.03)^t$
$\Rightarrow 2 = 1.03^t$
$t = \log_{1.03} 2 = 23.45 \simeq 23$ years (n.w.n)

Recurring decimals

Recurring decimals can be expressed as a sum to infinity of a geometric sequence, where the common ratio $r < 1$.

For example, $0.\dot{3} = 0.3333\ldots\ldots = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + \ldots\ldots$ (each term $\times \frac{1}{10}$ gives the next)

where $a = 0.3$ and $r = \frac{1}{10}$.

Similarly,

$$0.2\dot{3}\dot{5} = 0.2353535\ldots\ldots = 0.2 + [0.035 + 0.00035 + \ldots\ldots]$$
$$= 0.2 + \frac{35}{1000} + \frac{35}{100000} + \ldots\ldots$$
$$= 0.2 + \text{an infinite geometric series}$$

where $a = \frac{35}{1000}$ and $r = \frac{1}{100}$.

10. Write each of the following recurring decimals as an infinite geometric series. Hence express each as a decimal in the form $\frac{a}{b}$, $a, b \in N$.

(i) $0.\dot{7}$ (ii) $0.\dot{3}\dot{5}$ (iii) $0.2\dot{3}$ (iv) $0.\dot{3}7\dot{0}$ (v) $0.1\dot{6}\dot{2}$ (vi) $0.3\dot{2}\dot{1}$

Ex.4.5

(i) $0.\dot{7} = 0.7777\ldots = \frac{7}{10} + \frac{7}{100} + \frac{7}{1000} + \ldots$

$S_\infty = \frac{a}{1-r}$

$a = \frac{7}{10}$ $r = \frac{1}{10}$

$S_\infty = \frac{7/10}{1-1/10} = \frac{7/10}{9/10} = \frac{7}{9}$

(ii) $0.\dot{3}\dot{5} = 0.353535\ldots = \frac{35}{100} + \frac{35}{10000} + \frac{35}{1000000} + \ldots$

$a = 35/100$ $r = 1/100$

$S_\infty = \frac{a}{1-r}$

$S_\infty = \frac{(35/100)}{(1-1/100)} = \frac{(35/100)}{(99/100)} = \frac{35}{99}$

11. Find S_n, the sum to n terms, of $1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \ldots\ldots \left(\frac{1}{2}\right)^{n-1}$ and hence find S_∞, the sum to infinity of the series.
Find the least value of n such that $S_\infty - S_n < 0.001$.

$S_n = \frac{a(1-r^n)}{1-r}$

$a = 1$ $r = \frac{1}{2}$ $n = n$

$S_n = \frac{1(1-(\frac{1}{2})^n)}{1-1/2} = \frac{1(1-(\frac{1}{2})^n)}{1/2} = 2(1-(\frac{1}{2})^n)$

$S_\infty = \frac{a}{1-r}$

$S_\infty = \frac{1}{(1-\frac{1}{2})} = \frac{1}{(\frac{1}{2})} = 2$

$S_\infty - S_n < 0.001$

$\Rightarrow 2 - 2(1-(\frac{1}{2})^n) < 0.001$

Subtract 2

$-2(1-(\frac{1}{2})^n) < -1.999$

divide by -2 & change inequality

$1-(\frac{1}{2})^n > 0.9995$

Subtract 1

$-\frac{1}{2}^n > -0.0005$

change signs & inequality

$(\frac{1}{2})^n < 0.0005$

$\log_{\frac{1}{2}} 0.0005 = 10.965$

$\Rightarrow (\frac{1}{2})^{11} < 0.0005 \Rightarrow n = 11$

Understanding this question

S_8?

8. Evaluate $\sum_{r=1}^{8} 2 \times 3^r$.

T_n Rule

Sum from term 1 up to term 8

Solution

$S_8 = ?$

$T_n = 2(3^n)$

8. Evaluate $\sum_{r=1}^{8} 2 \times 3^r$.

$T_1 = 2(3^1) = 6$ $\times 3$

$T_2 = 2(3^2) = 18$

$T_3 = 2(3^3) = 54$

$a = 6$

$r = 3$

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$S_8 = \frac{6(1-3^8)}{1-3}$$

$$= 19,680$$

Section 4.6 Number Patterns

	Pattern	To find a
1st difference constant	$T_n = an + b$	a = 1st difference
2nd difference constant	$T_n = an^2 + bn + c$	$2a$ = 2nd difference
3rd difference constant	$T_n = an^3 + bn^2 + cn + d$	$6a$ = 3rd difference

Example 1

Express the nth term of the number pattern $-1, 13, 51, 125, 247, \ldots$ as a cubic polynomial.

	T_1	T_2	T_3		
	-1	13	51	125	247
1st D		14	38	74	122
2nd D			24	36	48
3rd D				12	12

$a = ?$

Cubic

$\Rightarrow$ 6a= 3rd difference

$6a = 12 \Rightarrow a = 2$

Shape: $an^3 + bn^2 + cn + d$

$\Rightarrow 2n^3 + bn^2 + cn + d$

get 3 equations to be able to solve for 3 unknowns b, c and d.

$T_1 = 2(1)^3 + b(1)^2 + c(1) + d = -1$

$2 + b + c + d = -1$

$b + c + d = -3$ ①

$T_2 = 2(2)^3 + b(2)^2 + c(2) + d = 13$

$16 + 4b + 2c + d = 13$

$4b + 2c + d = -3$ ②

$T_3 = 2(3)^3 + b(3)^2 + c(3) + d = 51$

$54 + 9b + 3c + d = 51$

$9b + 3c + d = -3$ ③

Solve

$b + c + d = -3$ ①
$4b + 2c + d = -3$ ②
$9b + 3c + d = -3$ ③

②-① $3b + c = 0$ ④ $\Rightarrow c = -3b$
③-① $8b + 2c = 0$ ⑤

④→⑤ $8b + 2(3b) = 0$
$8b - 6b = 0$
$2b = 0 \Rightarrow b = 0$

→④ $c = 3(0) \Rightarrow c = 0$

→① $0 + 0 + d = -3 \Rightarrow d = -3$

General expression $2n^3 + bn^2 + cn + d$

Cubic polynomial $= 2n^3 - 3$

6. Each of the following number patterns can be written in the form $an^3 + bn^2 + cn + d$. Find the values of a, b, c, and d in each case:

(i) 6, 27, 74, 159, 294
(ii) 3, −1, −1, 9, 35
(iii) −1, 2, 17, 50, 107

	T_1	T_2	T_3		
	6	27	74	159	294
Differences D_1		21	47	85	135
D_2			26	38	50
D_3				12	12

for cubic 6a= 3rd difference $a = ?$ $6a = 12 \Rightarrow a = 2$

cubic shape $an^3 + bn^2 + cn + d = 2n^3 + bn^2 + cn + d$

$T_1 =$ $2(1)^3 + b(1)^2 + c(1) + d = 6$
$2 + b + c + d = 6$
$b + c + d = 4$ ①

$T_2 =$ $2(2)^3 + b(2)^2 + c(2) + d = 27$
$16 + 4b + 2c + d = 27$
$4b + 2c + d = 11$ ②

$T_3 =$ $2(3)^3 + b(3)^2 + c(3) + d = 74$
$54 + 9b + 3c + d = 74$
$9b + 3c + d = 20$ ③

Solve

$b + c + d = 4$ ①
$4b + 2c + d = 11$ ②
$9b + 3c + d = 20$ ③

② − ① $3b + c = 7 \;=\; 3b + c = 7$ ④
③ − ① $8b + 2c = 16 \Rightarrow 4b + c = 8$ ⑤

⑤ − ④ $\Rightarrow b = 1$

→④ $\Rightarrow 3(1) + c = 7 \Rightarrow c = 4$

→① $\Rightarrow 1 + 4 + d = 4$
$5 + d = 4 \Rightarrow d = -1$

cubic shape $2n^3 + bn^2 + cn + d$

$= 2n^3 + n^2 + 4n - 1$ ✓

6. Each of the following number patterns can be written in the form $an^3 + bn^2 + cn + d$. Find the values of $a, b, c,$ and d in each case:

(i) 6, 27, 74, 159, 294
(ii) 3, −1, −1, 9, 35
(iii) −1, 2, 17, 50, 107

	T_1	T_2	T_3		
	3	−1	−1	9	35
Differences D1		−4	0	10	26
D2			4	10	16
D3				6	6

for cubic 6a= 3rd difference $a = ?$ $6a = 6 \Rightarrow a = 1$

cubic shape $an^3 + bn^2 + cn + d = n^3 + bn^2 + cn + d$

$T_1 =$ $(1)^3 + b(1)^2 + c(1) + d = 3$
$1 + b + c + d = 3$
$b + c + d = 2$ ①

$T_2 =$ $(2)^3 + b(2)^2 + c(2) + d = -1$
$8 + 4b + 2c + d = -1$
$4b + 2c + d = -9$ ②

$T_3 =$ $(3)^3 + b(3)^2 + c(3) + d = -1$
$27 + 9b + 3c + d = -1$
$9b + 3c + d = -28$ ③

Solve

$b + c + d = 2$ ①
$4b + 2c + d = -9$ ②
$9b + 3c + d = -28$ ③

② − ① $3b + c = -11$ ④
③ − ① $8b + 2c = -30 \Rightarrow 4b + c = -15$ ⑤

⑤ − ④ $\Rightarrow b = -4$

→ ④ $\Rightarrow 3(-4) + c = -11$
$-12 + c = -11 \Rightarrow c = 1$

→ ① $\Rightarrow -4 + 1 + d = 2$
$-3 + d = 2 \Rightarrow d = 5$

cubic shape $n^3 + bn^2 + cn + d$

$= n^3 - 4n^2 + n + 5$ ✓

6. Each of the following number patterns can be written in the form $an^3 + bn^2 + cn + d$. Find the values of a, b, c, and d in each case:

(i) 6, 27, 74, 159, 294
(ii) 3, −1, −1, 9, 35
(iii) −1, 2, 17, 50, 107

	T_1	T_2	T_3		
	−1	2	17	50	107
Differences D1		3	15	33	57
D2			12	18	24
D3				6	6

for cubic 6a= 3rd difference $a = ?$ $6a = 6 \Rightarrow a = 1$

cubic shape $an^3 + bn^2 + cn + d = n^3 + bn^2 + cn + d$

$T_1 =$ $(1)^3 + b(1)^2 + c(1) + d = -1$
$1 + b + c + d = -1$
$b + c + d = -2$ ①

$T_2 =$ $(2)^3 + b(2)^2 + c(2) + d = 2$
$8 + 4b + 2c + d = 2$
$4b + 2c + d = -6$ ②

$T_3 =$ $(3)^3 + b(3)^2 + c(3) + d = 17$
$27 + 9b + 3c + d = 17$
$9b + 3c + d = -10$ ③

Solve

$b + c + d = -2$ ①
$4b + 2c + d = -6$ ②
$9b + 3c + d = -10$ ③

② − ① $3b + c = -4$ ④
③ − ① $8b + 2c = -8 \Rightarrow 4b + c = -4$ ⑤

⑤ − ④ $\Rightarrow b = 0$

→④ $\Rightarrow 3(0) + c = -4 \Rightarrow c = -4$

→① $\Rightarrow 0 - 4 + d = -2$
$d = 2$

cubic shape $n^3 + bn^2 + cn + d$

$= n^3 - 4n + 2$ ✓