Example 3

In an arithmetic sequence, $T_4 = 6$ and $3T_2 = T_{10}$, find the values of a and d and hence write out the first 6 terms of the sequence.

$T_n = a + (n-1)d$

$T_4 = 6$

$\Rightarrow a + 3d = 6$

$3T_2 = T_{10}$

$3[a+d] = a + 9d$

$3a + 3d = a + 9d$

$2a = 6d$

$a = 3d$

Solve

Sub in

$3d + 3d = 6$

$6d = 6 \Rightarrow d = 1$

$a = 3(1) = 3$

$a = 3$

Example 4

If $p+2, 2p+3$ and $5p-2$ are three consecutive terms of an arithmetic sequence, find the value of $p, p \in R$.

$a = p + 2$

$d = T_2 - T_1$

$d = 2p + 3 - (p+2)$

$d = p + 1$

$d = T_3 - T_2$

$d = 5p - 2 - (2p+3)$

$d = 3p - 5$

$\Rightarrow 3p - 5 = p + 1$

$2p = 6$

$p = 3$

Sequence

1st 3 terms

$p+2, \ 2p+3, \ 5p-2 \ \ldots$

$= 3+2, \ 2(3)+3, \ 5(3)-2 \ \ldots$

$= 5, \ 9, \ 13, \ \ldots$

4. In an arithmetic sequence, $T_1 = 4$ and $T_7 = 22$. Using simultaneous equations, find
(i) the values of a and d (ii) the first five terms of the sequence (iii) T_{20}.

(i)

$a = ?$
$d = ?$

$T_1 = a = 4$ ✓

$T_7 = 22$

$T_n = a + (n-1)d$

$\Rightarrow 22 = 4 + (7-1)d$

-4: $18 = 6d$

$\div 6$: $3 = d$ ✓

First 5 Terms ? (ii)
Count up in 3's from 4

$4, 7, 10, 13, 16$ ✓

T_{20} ?
simplify rule for T_n

$T_n = 4 + (n-1)3$

$T_{20} = 4 + (19)3 = 61$ ✓

- Given an arithmetic sequence $T_1, T_2, T_3, T_4, T_5, \ldots\ldots T_n$,

$$T_3 - T_2 = T_4 - T_3 = T_5 - T_4 = \text{the common difference } (d).$$

In general terms:

$$T_{n+1} - T_n = d \text{ (the common difference).}$$

A corollary to this is as follows:

To prove that a sequence is arithmetic, we must show that $T_{n+1} - T_n$ is a constant.

- Also, if $T_{n+1} - T_n > 0$, then the sequence is increasing

if $T_{n+1} - T_n < 0$, then the sequence is decreasing.

Note, to find T_{n+1}, substitute $(n + 1)$ for n in T_n.

If $T_n = 3n + 1$,

$T_{n+1} = 3(n + 1) + 1 = 3n + 4$.

Example 5

Given (i) $T_n = \frac{n+1}{2}$

(ii) $T_n = \frac{2}{n+1}$, determine whether

(a) the sequence is arithmetic or not

(b) the sequence is increasing or decreasing.

Is $T_{n+1} - T_n = \text{const.}$?

(i) $T_{n+1} = \frac{n+1+1}{2} = \frac{n+2}{2}$

$\frac{n+2}{2} - \left(\frac{n+1}{2}\right) = \frac{\cancel{n}+2-\cancel{n}-1}{2} = \frac{1}{2}$

$\frac{1}{2}$ is a constant $\Rightarrow$ Increasing and arithmetic

(ii) $T_n = \frac{2}{n+1} \Rightarrow T_{n+1} = \frac{2}{n+1+1} = \frac{2}{n+2}$

$T_{n+1} - T_n = \frac{2}{n+2} - \frac{2}{n+1} = \frac{2(n+1) - 2(n+2)}{(n+2)(n+1)}$

$= \frac{\cancel{2n}+2-\cancel{2n}-4}{(n+2)(n+1)} = \frac{-2}{(n+2)(n+1)}$ not a constant $\Rightarrow$ not arithmetic

6. In an arithmetic sequence, $T_{13} = 27$ and $T_7 = 3T_2$. Find expressions in terms of n for T_{13}, T_7 and T_2 and hence find the values of a and d.
Write down the first six terms of the sequence.

$T_n = a + (n-1)d$

$T_{13} = a + (13-1)d \Rightarrow T_{13} = a + 12d$

$T_7 = a + (7-1)d \Rightarrow T_7 = a + 6d$

$T_2 = a + (2-1)d \Rightarrow T_2 = a + d$

$T_{13} = 27 \Rightarrow$ $a + 12d = 27$ ①

$T_7 = 3T_2 \Rightarrow$ $a + 6d = 3(a+d)$

$a + 6d = 3a + 3d$

$3d = 2a \Rightarrow d = \frac{2}{3}a$ ②

② → ① $a + 12\left(\frac{2}{3}a\right) = 27$

$a + 8a = 27$

$9a = 27 \Rightarrow a = 3$ ③

③ → ② $d = \frac{2}{3}(3) \Rightarrow d = 2$

7. (i) If $2k + 2, 5k - 3$ and $6k$ are three consecutive terms of an arithmetic sequence, find the value of $k, k \in \mathbb{Z}$.

(ii) Given that $4p, -3 - p$ and $5p + 16$ are three consecutive terms of an arithmetic sequence, find the value of $p, p \in \mathbb{Z}$.

(i) $T_n - T_{n-1} = d$

If you subtract subsequent terms you at way get d

$$\Rightarrow (5k-3)-(2k+2) = (6k)-(5k-3)$$
$$5k-3-2k-2 = 6k-5k+3$$
$$3k-5 = k+3$$
$$2k = 8$$
$$k = 4$$

check: $2(4)+2,\ 5(4)-3,\ 6(4)$

$10,\ 17,\ 24 \ldots$

This is an arithmetic series, $d=7$

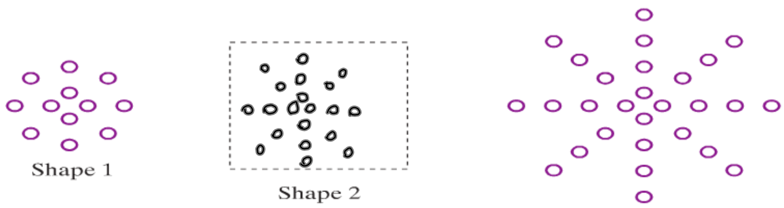
(ii)

$$\Rightarrow (-3-p)-(4p) = (5p+16)-(-3-p)$$
$$-3-p-4p = 5p+16+3+p$$
$$-3-5p = 6p+19$$
$$-11p = 22$$
$$p = -2$$

check: $4(-2),\ -3-(-2),\ 5(-2)+16$

$-8,\ -1,\ 6$ $\Rightarrow$ series $d=7$

8.



Shape 1 Shape 2 Shape 3

Three shapes were drawn on a wall.
The second shape was removed accidentally. Given that the shapes were drawn in arithmetic sequence, draw shape 2. ✓

(i) Write a number sequence for the number of circles used in each shape and hence find T_n for the sequence.

(ii) How many circles are needed for shape 15?

(iii) Which shape requires 164 circles?

(i) $12,\ 20,\ 28$

$T_n = a + (n-1)d$

$$T_n = 12 + (n-1)8$$
$$T_n = 12 + 8n - 8$$
$$T_n = 4 + 8n$$

(ii) $T_{15} = 4 + 8(15) = 124$

$n: ?$ (iii) $164 = 4 + 8n \Rightarrow 160 = 8n$

$n = 20$

The Great Gauss Summation Trick

One of the most famous mathematicians of all times was named Karl Gauss. One day, as the story goes, his teacher gave the class an assignment to keep them busy so that he could take a nap in the back of the class. The problem he assigned would keep most of us busy for at least a half an hour, if not more. However, to his teacher's surprise, young Mr. Gauss solved it in seconds.

Here is the problem the teacher assigned. Students were told to add all the whole numbers from one to one hundred. That is, 1+2+3+4+5 …98+99+100. In less time than it took most students to write out this one hundred number addition problem, Gauss got the answer. The sum is 5,050 he told his teacher confidently, and so it was. But how did he arrive at this answer in so short a time?

Gauss was a genius, and geniuses sometimes see things differently than most of us non genius types. But that doesn't mean that after being shown the way that we can not solve a problem like a genius would, having first been shown the way. Here is how young Gauss arrived at his answer so quickly. He observed that in the series of numbers 1 +2 + 3 +4 …97 + 98+ 99 + 100, the sum of pairs of numbers from each end, and working in toward the middle summed to the same value,101. In other words, 1 + 100, 2 +99, 3 + 98, 4 + 97 etc. all sum to 101! Since there are fifty pair of numbers in the series 1 to 100, Gauss reasoned that the sum of all the numbers would be 50 times 101 or 5,050.

$$S_n = \frac{(T_1 + T_n)n}{2}$$

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$$

Exercise 4.3

1. Find S_n and S_{20} of each of the following arithmetic sequences:

(i) $1 + 5 + 9 + 13 + \ldots\ldots$ (ii) $50 + 48 + 46 + 44 + \ldots\ldots$
(iii) $1 + 1.1 + 1.2 + 1.3 \ldots\ldots$ (iv) $-7 - 3 + 1 + 5 + \ldots\ldots$

$S_n = \frac{n}{2}[2a + (n-1)d]$

(i) $a = 1$, $d = 4$

$S_n = \frac{n}{2}[2(1) + (n-1)4] = \frac{n}{2}[2 + 4n - 4]$

$S_n = \frac{n}{2}[4n - 2] = n[2n - 1] = 2n^2 - n$

$S_{20} = 2(20)^2 - (20) = 780$ ✓

$S_n = \frac{n}{2}[2a + (n-1)d]$

(ii) $a = 50$, $d = -2$

$S_n = \frac{n}{2}[2(50) + (n-1)(-2)] = \frac{n}{2}[100 - 2n + 2]$

$S_n = \frac{n}{2}[102 - 2n] = n[51 - n] = 51n - n^2$

$S_{20} = 51(20) - (20)^2 = 620$

Exercise 4.3

1. Find S_n and S_{20} of each of the following arithmetic sequences:

(i) $1 + 5 + 9 + 13 + ...$ (ii) $50 + 48 + 46 + 44 + ...$
(iii) $1 + 1.1 + 1.2 + 1:3 .....$ (iv) $-7 - 3 + 1 + 5 + ...$

$S_n = \frac{n}{2}[2a + (n-1)d]$

(iii) $a = 1, \quad d = 0.1$

$$S_n = \frac{n}{2}[2(1) + (n-1)(0.1)] = \frac{n}{2}[2 + 0.1n - 0.1]$$

$$S_n = \frac{n}{2}[1.9 + 0.1n] = \frac{19n + n^2}{20}$$

$$S_{20} = \frac{19(20) + (20)^2}{20} = 19 + 20 = 39 \checkmark$$

$S_n = \frac{n}{2}[2a + (n-1)d]$

(iv) $a = -7, \; d = 4$

$$S_n = \frac{n}{2}[2(-7) + (n-1)(4)] = \frac{n}{2}[-14 + 4n - 4]$$

$$S_n = \frac{n}{2}[4n - 18] = n[2n - 9] = 2n^2 - 9n$$

$$S_{20} = 2(20)^2 - 9(20) = 620 \checkmark$$

Example 1

Find the sum of the series $4 + 11 + 18 + 25 + ... + 144$.

$+7$

$a = 4 \qquad d = 7 \qquad n = ? \qquad T_n = 144$

$T_n = a + (n-1)d$

$$4 + (n-1)7 = 144$$
$$4 + 7n - 7 = 144$$
$$7n - 3 = 144$$
$$7n = 147$$
$$n = 21$$

$S_n = \frac{n}{2}[2a + (n-1)d]$

$$S_{21} = \frac{21}{2}[2(4) + (21-1)7]$$
$$= \frac{21}{2}[8 + (20)7]$$
$$= \frac{21}{2}[148]$$
$$= 1554$$

2. Find the sum of each of the following:

(i) $6 + 10 + 14 + 18 + \ldots.. 50$ (ii) $1 + 2 + 3 + 4 + \ldots.. 100$

(iii) $80 + 74 + 68 + 62 \ldots.. -34$

(i)

In these question we first need to discover which term is the last one.

$T_n = a + (n-1)d$

$a = 6, \quad d = 4, \quad T_n = 50$

$n = ?$

$\Rightarrow \quad 50 = 6 + (n-1)4$

$50 = 6 + 4n - 4$

$50 = 2 + 4n$

$48 = 4n$

$12 = n$

$S_n = \frac{n}{2}[2a + (n-1)d]$

$\Rightarrow \quad S_{12} = \frac{12}{2}[2(6) + (12-1)4]$

$= 6[12 + 44] = 6[56]$

$S_{12} = 336$ ✓

2. Find the sum of each of the following:

(i) $6 + 10 + 14 + 18 + \ldots.. 50$ (ii) $1 + 2 + 3 + 4 + \ldots.. 100$

(iii) $80 + 74 + 68 + 62 \ldots.. -34$

(ii)

In these question we first need to discover which term is the last one.

$T_n = a + (n-1)d$

$a = 1, \quad d = 1, \quad T_n = 100$

$n = ?$

$\Rightarrow \quad 100 = 1 + (n-1)1$

$100 = 1 + n - 1$

$100 = n$

$S_n = \frac{n}{2}[2a + (n-1)d]$

$S_{100} = \frac{100}{2}[2(1) + (100-1)1]$

$= 50[2 + 99] = 50[101]$

$S_{100} = 5050$ ✓

2. Find the sum of each of the following:

(i) $6 + 10 + 14 + 18 + \ldots\ldots 50$ (ii) $1 + 2 + 3 + 4 + \ldots\ldots 100$

(iii) $80 + 74 + 68 + 62 \ldots\ldots -34$

In these question we first need to discover which term is the last one.

$T_n = a + (n-1)d$

$n = ?$

$a = 80,\ d = -6,\ T_n = -34$

$$\Rightarrow -34 = 80 + (n-1)(-6)$$
$$-34 = 80 - 6n + 6$$
$$-34 = 86 - 6n$$
$$-120 = -6n$$
$$20 = n$$

$S_n = \frac{n}{2}[2a + (n-1)d]$

$$S_{20} = \frac{20}{2}[2(80) + (20-1)(-6)]$$
$$= 10[160 + 19(-6)]$$
$$= 10[160 - 114] = 10[46]$$
$$S_{20} = 460 \checkmark$$