The Circle and The Line

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1st December 2011

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Announcements

- If anyone has emailed Kieran for last week's notes, they will be sent out on either Monday or Tuesday of next week.

- If there are any new people here, on the sign in sheet is the room number (eg. C1-061) you must attend for the workshop. Please note this room number as you MUST attend this room.

New Syllabus

2.2 Co-ordinate geometry	– use slopes to show that two lines are • parallel • perpendicular	– calculate the area of a triangle – recognise the fact that the relationships $y= mx+c$, $y-y_1 = m (x- x_1)$ and $ax + by + c = 0$ are linear – solve problems involving slopes of lines – recognise that $(x-h)^2 + (y-k)^2 = r^2$ represents the relationship between the x and y co-ordinates of points on a circle centre (h, k) and radius r – solve problems involving a line and a circle with centre (0, 0)	– solve problems involving • the perpendicular distance from a point to a line • the angle between two lines – divide a line segment in a given ratio m:n – recognise that $x^2+y^2 +2gx+2fy+c = 0$ represents the relationship between the x and y co-ordinates of points on a circle centre (-g,-f) and radius r where $r = \sqrt{(g^2+f^2 -c)}$ – solve problems involving a line and a circle

Previous Question Concepts – The Line

- 2011

  - Division of a line segment in a given ratio

  - Linear transformations

  - Area of a triangle

  - ***Theorem:*** Proof that *f* maps every pair of parallel lines to a pair of parallel lines.

- 2010

  - Perpendicular lines

  - Calculating points where a line crosses x and y-axis

  - Area of a triangle

  - Linear transformations: Find the image of a line

  - Angle between 2 lines

Previous Question Concepts – The Line

- 2009

  - Point of intersection of two lines

  - Equation of a line

  - ***Theorem:*** Proof of measure of angle between two lines with slopes m_1 and m_2

  - Extension questions using this theorem

  - Linear transformations: Prove $f(l)$ is a line

- 2008

  - Parametric & Cartesian equations of a line

  - Distance between points

  - Linear transformations: finding the image of different points

  - ***Theorem:*** Perpendicular distance between a point and a line

Previous Question Concepts – The Line

- 2007

  - Area of a triangle

  - Linear transformations: Images and ratios

  - Point of intersection of two lines

  - Slope

- 2006

  - Perpendicular lines

  - Division of a line segment

  - ***Theorem:*** Proof of measure of angle between two lines with slopes m_1 and m_2

  - Theorem application question

Summary of Concepts Checklist – The Line

Concept	√ or x
Division of a line in a given ratio	
Area of a triangle	
Angle between 2 lines	
Concurrencies of a triangle	
Perpendicular distance from a point to a line	
Parametric equations of a line	
Linear transformations	
Theorems	

Division of a line in a given ratio

- P(x_1, y_1) and Q $(x_2, y_2) \to [PQ]$ can be divided internally or externally.
- **Internal division:**

If R divides [PQ] internally in the ratio m:n

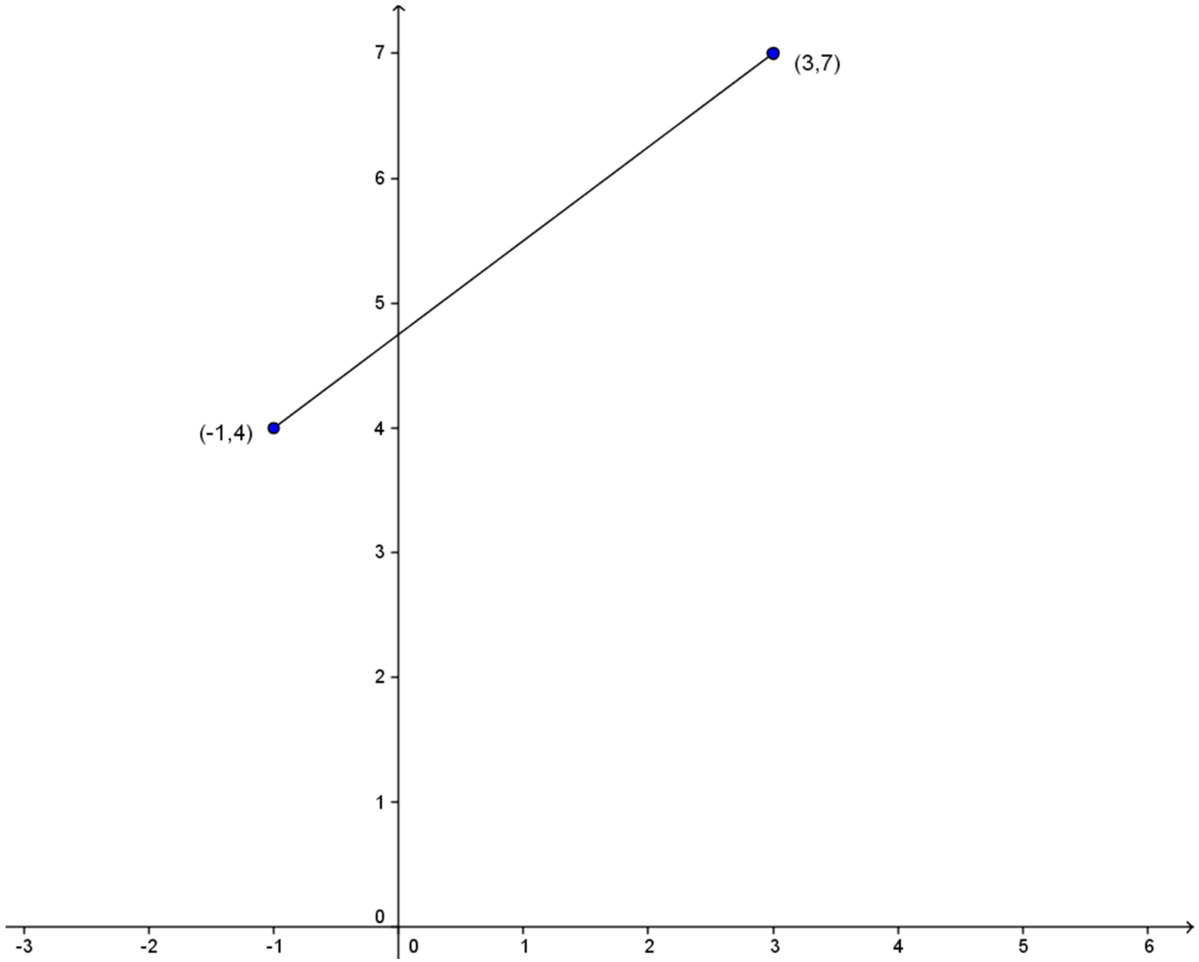
$$R = \left(\frac{nx_1 + mx_2}{m+n}, \frac{ny_1 + my_2}{m+n}\right)$$

Pg18 of tables

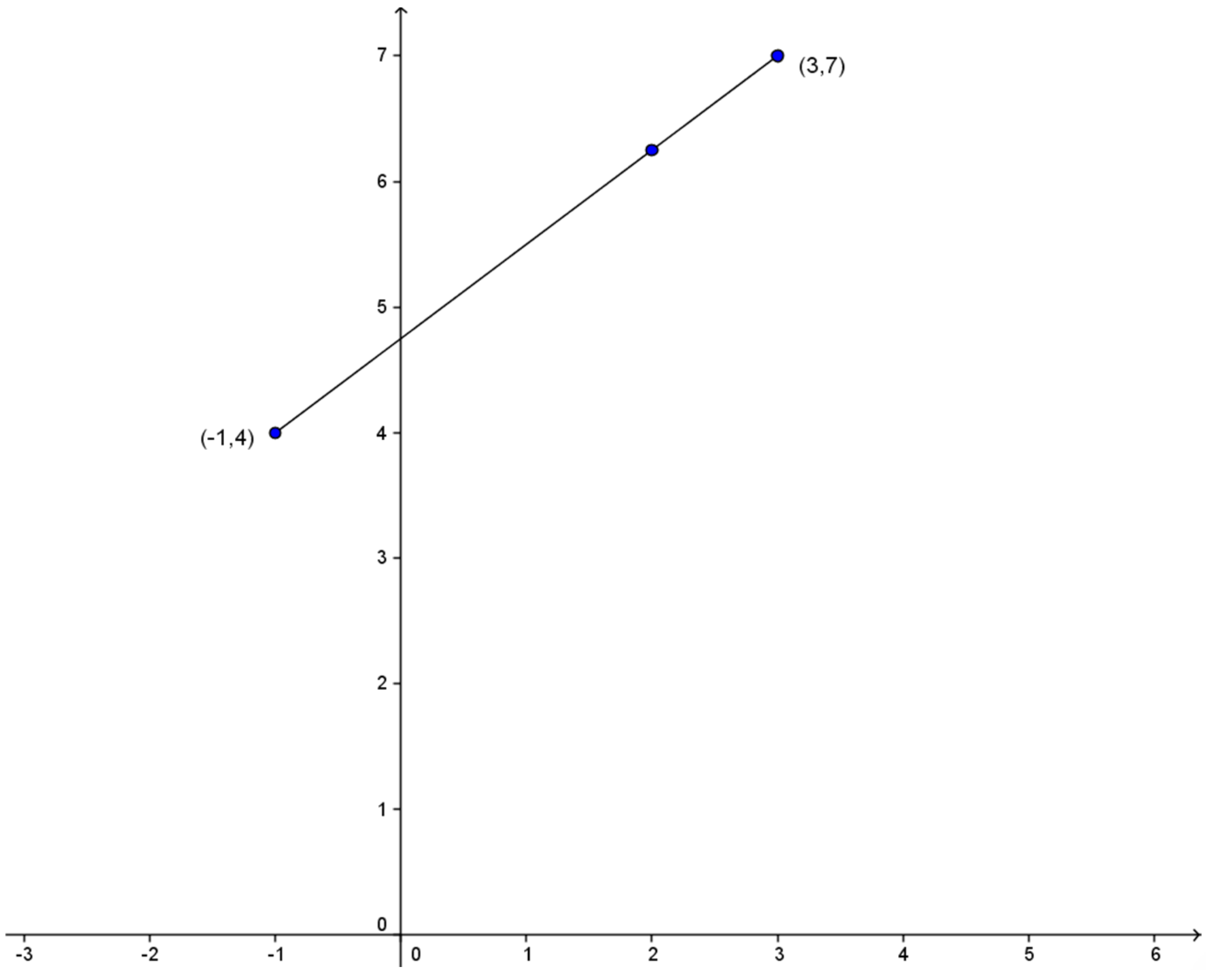
The Line - 2011

- (a) P and Q are the points $(-1,4)$ and $(3,7)$ respectively. Find the co-ordinates of the point that divides $[PQ]$ internally in the ratio $3:1$.
- $x_1 = -1;\ x_2 = 3;\ y_1 = 4;\ \ y_2 = 7$
- m=3; n=1
- $x = \frac{3(3)+1(-1)}{3+1}$, $y = \frac{3(7)+1(4)}{3+1}$
- $x = \frac{8}{4}$, $y = \frac{25}{4}$
- $\left(2, \frac{25}{4}\right)$

2011 Q3(a)



2011 Q3(a)



Angle between 2 lines

- If two lines p and q have slopes m_1 and m_2 respectively, and θ is the angle between them, then:

$$\tan\theta = \pm\frac{m_1 - m_2}{1 + m_1 m_2}$$

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The Line 2009 (b)

- (i) Prove the formula
- (ii) Find the equations of the two lines that pass through the point (6,1) and make an angle of $45°$ with the line $x + 2y = 0$.

(ii) Line: $x + 2y = 0 \;\Rightarrow\; m_1 = -\frac{1}{2}$

$$\theta = 45°$$

$$\tan 45° = \pm\left(\frac{-\frac{1}{2} - m_2}{1 + \left(-\frac{1}{2}\right) m_2}\right)$$

$$\pm 1 = \frac{1 + 2m_2}{-2 + m_2}$$

2009 (b)

$$1 = \frac{1 + 2m_2}{-2 + m_2}$$

$$-2 + m_2 = 1 + 2m_2$$

$$m_2 = -3$$

Equation of l_1:

$(6,1)$; $m = -3$

$3x + y + k = 0$

$3(6) + 1 + k = 0$

$k = -19$

$l_1 : 3x + y - 19 = 0$

$$-1 = \frac{1 + 2m_2}{-2 + m_2}$$

$$2 - m_2 = 1 + 2m_2$$

$$m_2 = \frac{1}{3}$$

Equation of l_2:

$(6,1)$; $m = \frac{1}{3}$

$$x - 3y + k = 0$$

$$6 - 3(1) + k = 0$$

$$k = -3$$

$$l_2 : x - 3y - 3 = 0$$

Parametric Equations

- Parametric equations are simply a representation in a different variable or number of variables (in leaving cert you only have to deal with one extra variable, usually t).

- You know that a line is given in the form $ax + by + c = 0$.

- All that a parametric equation does is give us an x equation which defines the x co-ordinates you are plotting, and a y equation which defines the y co-ordinates.

Parametric Equations 2008 (a)

- The parametric equations $x = 7t - 4$ and $y = 3 - 3t$ represents a line, where $t \in \mathbb{R}$. Find the Cartesian equation of the line.

- $x = 7t - 4 \Rightarrow \quad t = \frac{x+4}{7}$
- $y = 3 - 3t \Rightarrow \quad t = \frac{3-y}{3}$
- $\Rightarrow \frac{x+4}{7} = \frac{3-y}{3}$
- $3x + 12 = 21 - 7y$
- $3x + 7y - 9 = 0$

What is a linear transformation?

- People may be aware HOW to solve a linear transformation however you MAY not be certain about what they actually are.
- Just like central symmetry, axial symmetry and translations, linear combinations move points from one place to another. However, linear combinations use both the x and y value of the point being moved and sub them into a function involving x and y. For instance a line is mapped to a line.
- Distance, area and perpendicularity are three things which may not be conserved under linear transformations. (i.e. if perpendicularity is the case before the transformation, it may not exist after the transformation)

The Line 2010

- (a) The line $3x + 4y - 7 = 0$ is perpendicular to the line $ax - 6y - 1 = 0$. Find the value of a.

- (b) (i) The line $4x - 5y + k = 0$ cuts the x-axis at P and the y-axis at Q. Write down the co-ordinates of P and Q in terms of k.

  (ii) The area of the triangle OPQ is 10 square units, where O is the origin. Find the two possible values of k.

- (c) f is the transformation $(x, y) \rightarrow (x', y')$, where $x' = x + y$ and $y' = x - y$. The line l has equation $y = mx + c$.

  (i) Find the equation of $f(l)$, the image of l under f.

  (ii) Find the value(s) of m for which $f(l)$ makes an angle of $45°$ with l.

Previous Question Concepts – The Circle

- 2011
  - Parametric and Cartesian Equations
  - Equation of a circle passing through 3 points
  - Intersection of circles
  - Tangents at points of intersection
- 2010
  - Equation of a circle with centre on point on circle
  - Find centre and radius when given equation
  - Tangent to a circle

Previous Question Concepts – The Circle

- 2009

  - Show a point lies on a circle when given equation

  - Equation of a tangent

  - Equations of 2 circles when given points of intersection and distance to a common chord.

- Other years require similar knowledge and understanding.

Summary of Concepts Checklist – The Circle

Concept	√ or x
Equation of a circle (in various forms)	√ (2011)
Parametric equations of a circle	√ (2011)
Proving that a locus is a circle	
Intersection of a line and circle	
Finding the equation of a circle (geometric/algebraic approaches)	
Proving a line is a tangent to a circle	
Tangents parallel, or perpendicular, to a given line	
Equations of tangents from a point outside the circle	
Touching/Intersecting circles	√ (2011)
Chords	

Tips

- Sketch whatever information the question is giving you. Visualising the problem is putting you far closer to solving it than if you just learn off steps!

- Think about how you might solve the problem once you have set it up visually (using formulae etc.)

- Try it out.

When attempting these questions in class or in your study time ALWAYS try to visualise, sketch, and problem solve (DON'T try to learn off steps!!!)

The Circle 2011

- (a) The following parametric equations define a circle:

  $$x = 2 + 3\sin\theta\,,\ \ y = 3\cos\theta \text{ where } \theta \in \mathbb{R}$$

  What is the Cartesian equation of the circle?

- (b) Find the equation of the circle that passes through the points $(0,3)$, $(2,1)$ and $(6,5)$.

- (c) The circle c_1: $x^2 + y^2 - 8x + 2y - 23 = 0$ has centre A and radius r_1.

  The circle c_2: $x^2 + y^2 + 6x + 4y + 3 = 0$ has centre B and radius r_2.

  (i) Show that c_1 and c_2 intersect at two points.

  (ii) Show that the tangents to c_1 at these points of intersection pass through the centre of c_2.

The Circle 2011 (a)

- Parametric equations again. This time trigonometric functions are given. We know we need the Cartesian form so we must find a way of manipulating the trigonometric parts in order to get just x’s, y’s and numbers.

- $x = 2 + 3\sin\theta$
- $\Rightarrow x - 2 = 3\sin\theta$
- $y = 3\cos\theta$

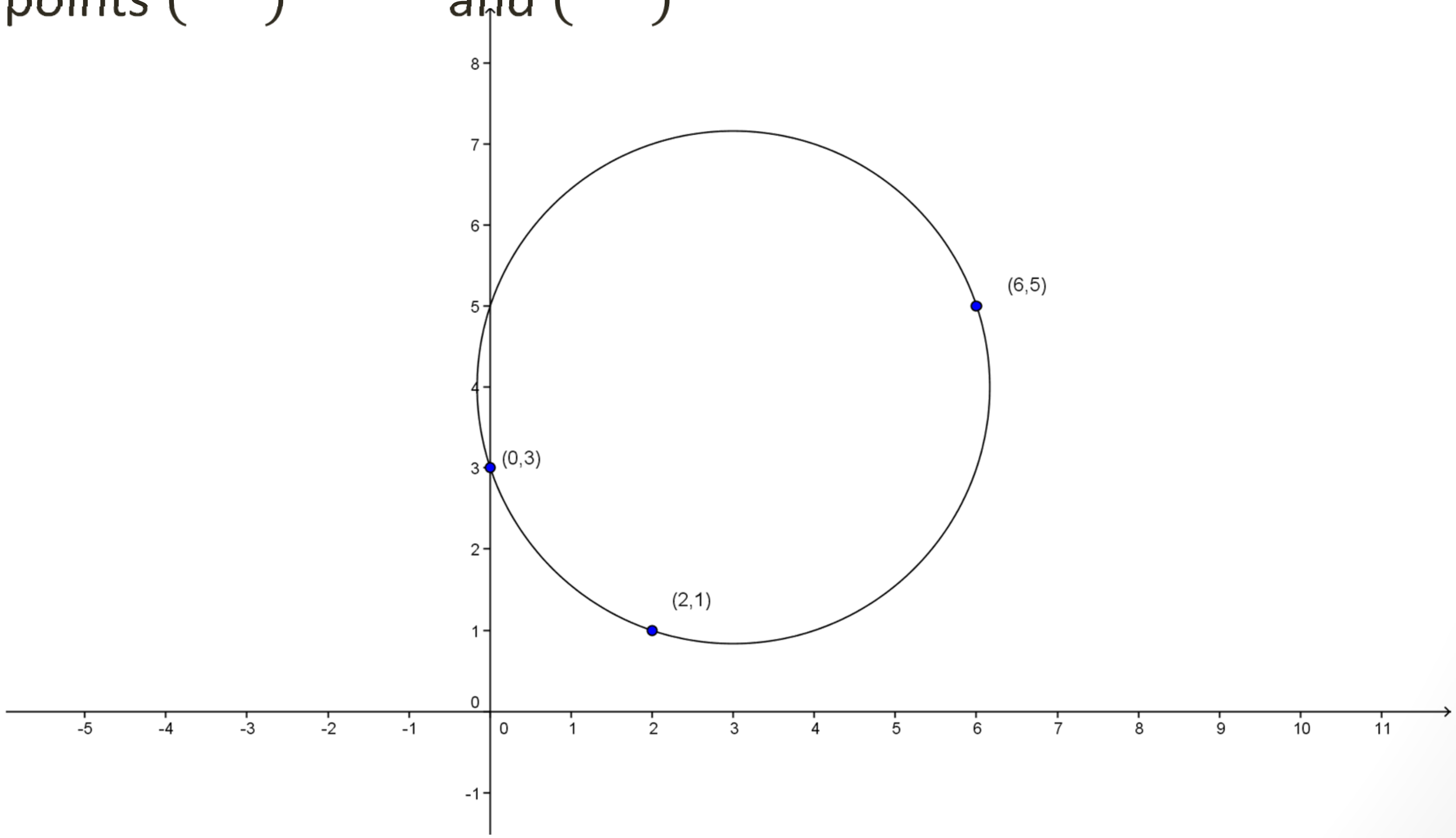
- $\Rightarrow (x-2)^2 + y^2 = 9sin^2\theta + 9cos^2\theta$

- $(x-2)^2 + y^2 = 9(sin^2\theta + cos^2\theta)$

- $(x-2)^2 + y^2 = 9$

The Circle 2011

- (b)Find the equation of the circle that passes through the points (   )   and (   )



The Circle 2011 (b)

- We know that each of the three points given are on the circle. Therefore when we sub them into the general equation of a circle we will get zero as our answer.
- General Equation: $x^2 + y^2 + 2gx + 2fy + c = 0$

- $(0,3) \in c \rightarrow (0)^2 + (3)^2 + 2g(0) + 2f(3) + c = 0$

  $9 + 6f + c = 0$

  $6f + c = -9 \longleftarrow 1$
- $(2,1) \in c \rightarrow (2)^2 + (1)^2 + 2g(2) + 2f(1) + c = 0$

  $4g + 2f + c = -5 \longleftarrow 2$
- $(6,5) \in c \rightarrow (6)^2 + (5)^2 + 2g(6) + 2f(5) + c = 0$

  $12g + 10f + c = -61 \longleftarrow 3$

- We must now solve for g,f and c.
- 3 – 2
- $12g + 10f + c = -61$
- $\underline{-4g - 2f - c \quad = \quad 5}$
- $8g + 8f = -56 \quad \Rightarrow g + f = -7$ ⟵ 4

- 1 – 2
- $6f + c = -9$
- $\underline{-4g - 2f - c = 5}$
- $-4g + 4f = -4 \quad \Rightarrow \quad -g + f = -1$ ⟵ 5

- 4 + 5
- $g + f = -7$
- $\underline{-g + f = -1}$
- $2f = -8 \quad \Rightarrow f = -4$

- $f = -4$

- $g + (-4) = -7 \quad \Rightarrow g = -3$

- $6(-4) + c = -9 \quad \Rightarrow c = 15$

- Equation of the circle is therefore:
- c: $x^2 + y^2 + 2gx + 2fy + c = 0$

- $x^2 + y^2 + 2(-3)x + 2(-4)y + 15 = 0$

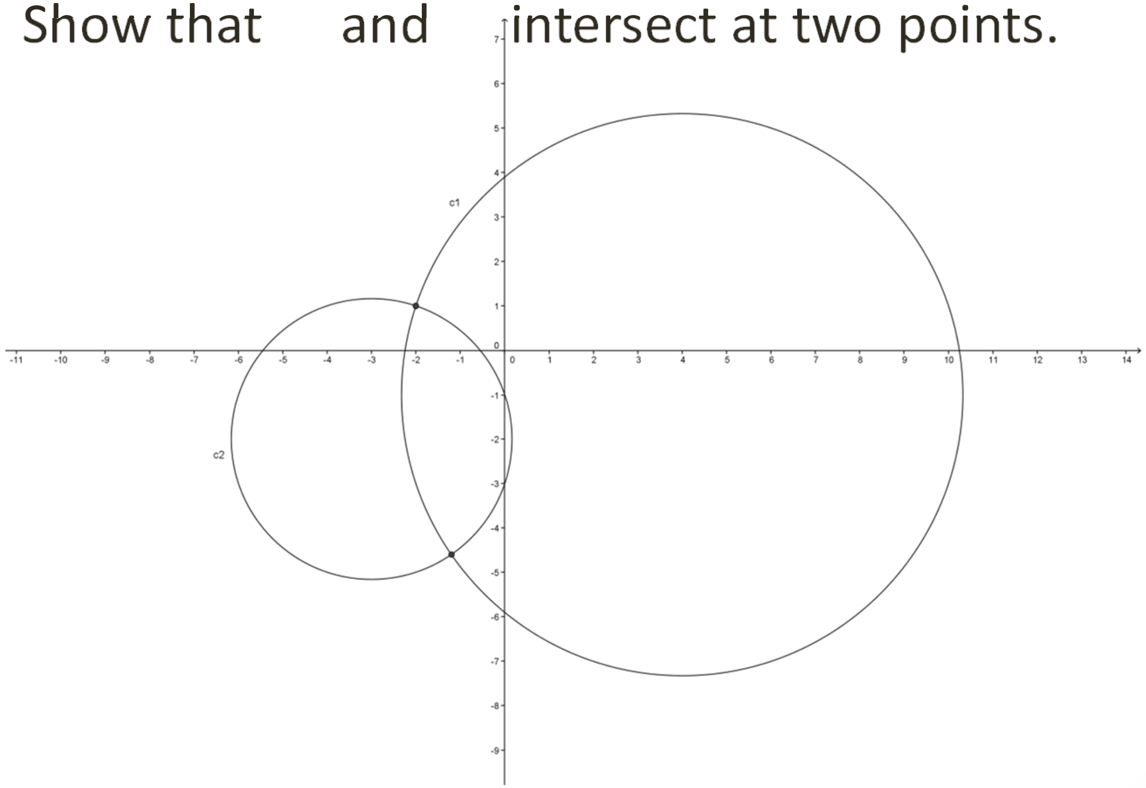
- $x^2 + y^2 - 6x - 8y + 15 = 0$

The Circle 2011

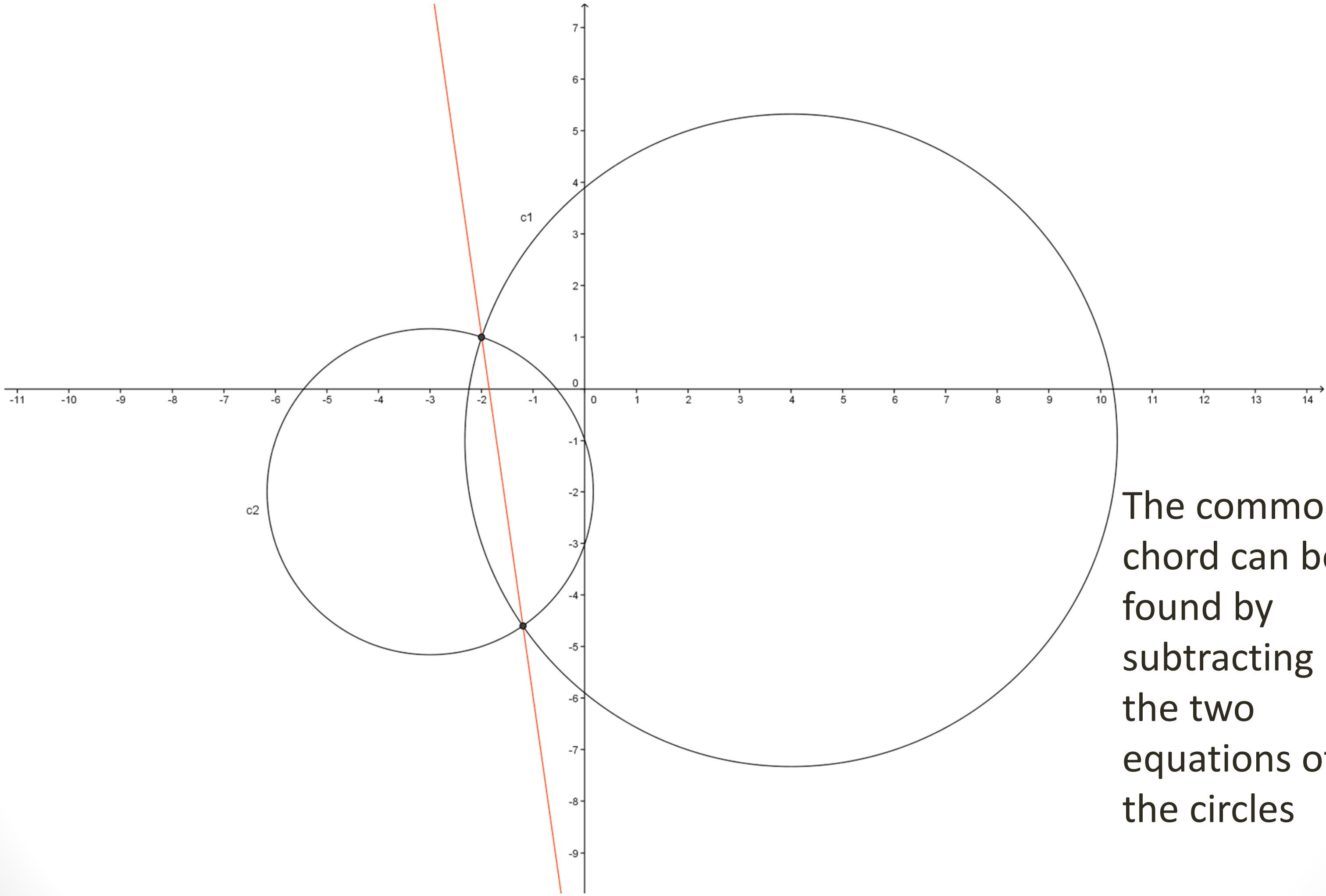
- (c) The circle has centre and radius .

  The circle has centre B and radius .

  (i) Show that and intersect at two points.



2011 Q1(c) (i)



The common chord can be found by subtracting the two equations of the circles

2011 Q1 (c) (i)

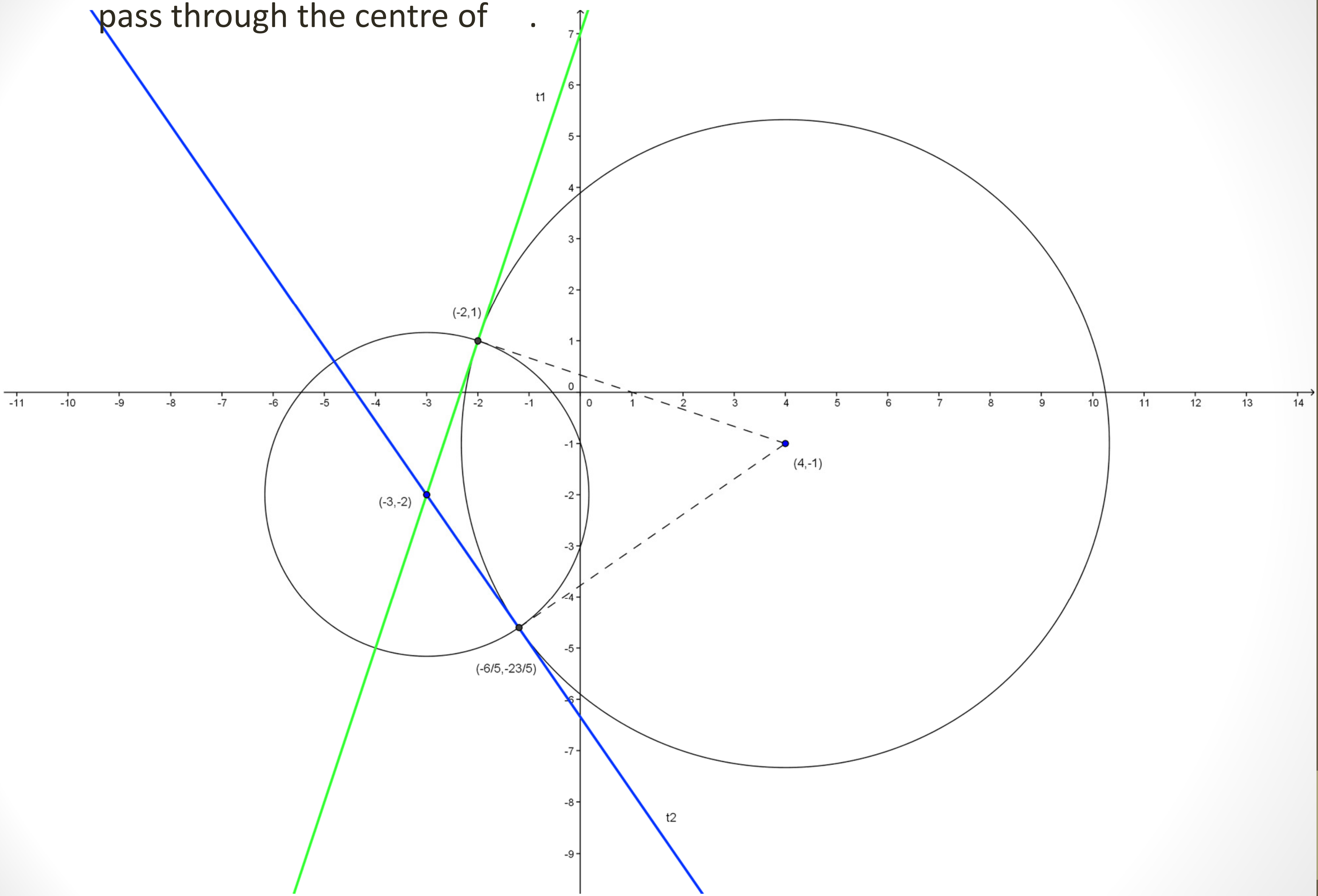
- Common chord:
- $x^2 + y^2 - 8x + 2y - 23 = 0$
- $\underline{-x^2 - y^2 - 6x - 4y - 3 = 0}$
- $-14x - 2y - 26 = 0$
- $\Rightarrow \quad 7x + y + 13 = 0$

- $y = -7x - 13$
- This is the equation of the common chord

2011 Q1 (c) (i)

- So now we have:
- $x^2 + y^2 + 6x + 4y + 3 = 0$
- $x^2 + (-7x - 13)^2 + 6x + 4(-7x - 13) + 3 = 0$
- $x^2 + 49x^2 + 182x + 169 + 6x - 28x - 52 + 3 = 0$
- $50x^2 + 160x + 120 = 0$
- $5x^2 + 16x + 12 = 0$
- $(5x + 6)(x + 2) = 0$
- $x = -\frac{6}{5} \quad , \; x = -2$
- $@x = -\frac{6}{5}, y = -7\left(-\frac{6}{5}\right) - 13 = -\frac{23}{5}$
  $\Rightarrow p.o.i\ is\ \left(-\frac{6}{5}, -\frac{23}{5}\right)$
- $@x = -2, \;\; y = -7(-2) - 13 = 1 \;\Rightarrow p.o.i\ is\ (-2,1)$

2011 Q1 (c) (ii) Show that the tangents to at these points of intersection pass through the centre of .



2011 Q1(c)(ii)

- Equation of t_1 (green tangent):
- Find the slope between point of contact (-2,1) and the centre of c_1 (4,-1).
- $m_1 = \frac{1-(-1)}{-2-4} = \frac{2}{-6} = -\frac{1}{3}$
- The slope of t_1 is perpendicular to this slope $\Rightarrow m^{\perp} = 3$
- Equation of t_1: $3x - y + k = 0$
- We know that $(-2,1) \in t_1$
- $\Rightarrow 3(-2) - 1 + k = 0, \quad so\ k = 7$
- $t_1: 3x - y + 7 = 0$

2011 Q1 (c)(ii)

- Going back to the question is the centre of c_2 on this line?
- Centre of $c_2 = (-3, -2)$
- $3(-3) - (-2) + 7 = 0$
- $0 = 0$
- $\Rightarrow t_1$ passes through the centre of c_2.

- Now we must try this for the other tangent.

2011 Q1(c)(ii)

- Equation of t_2 (blue tangent):
- Find the slope between point of contact $\left(-\frac{6}{5}, -\frac{23}{5}\right)$ and the centre of c_1 (4,-1).
- $m_1 = \frac{-\frac{23}{5}-(-1)}{-\frac{6}{5}-4} = \frac{9}{13}$
- The slope of t_2 is perpendicular to this slope $\Rightarrow m^{\perp} = -\frac{13}{9}$
- Equation of t_2: $13x + 9y + k = 0$
- We know that $\left(-\frac{6}{5}, -\frac{23}{5}\right) \in t_2$
- $\Rightarrow 13\left(-\frac{6}{5}\right) + 9\left(-\frac{23}{5}\right) + k = 0, \quad so\ k = 57$
- $t_2: 13x + 9y + 57 = 0$

2011 Q1(c)(ii)

- Finally is the centre of c_2 $(-3,-2)$ on this line?
- $13(-3)+9(-2)+57=0$
- $0=0$
- $\Rightarrow t_2$ passes through the centre of c_2.
- So the tangents to c_1 at the points of intersection pass through the centre of c_2.

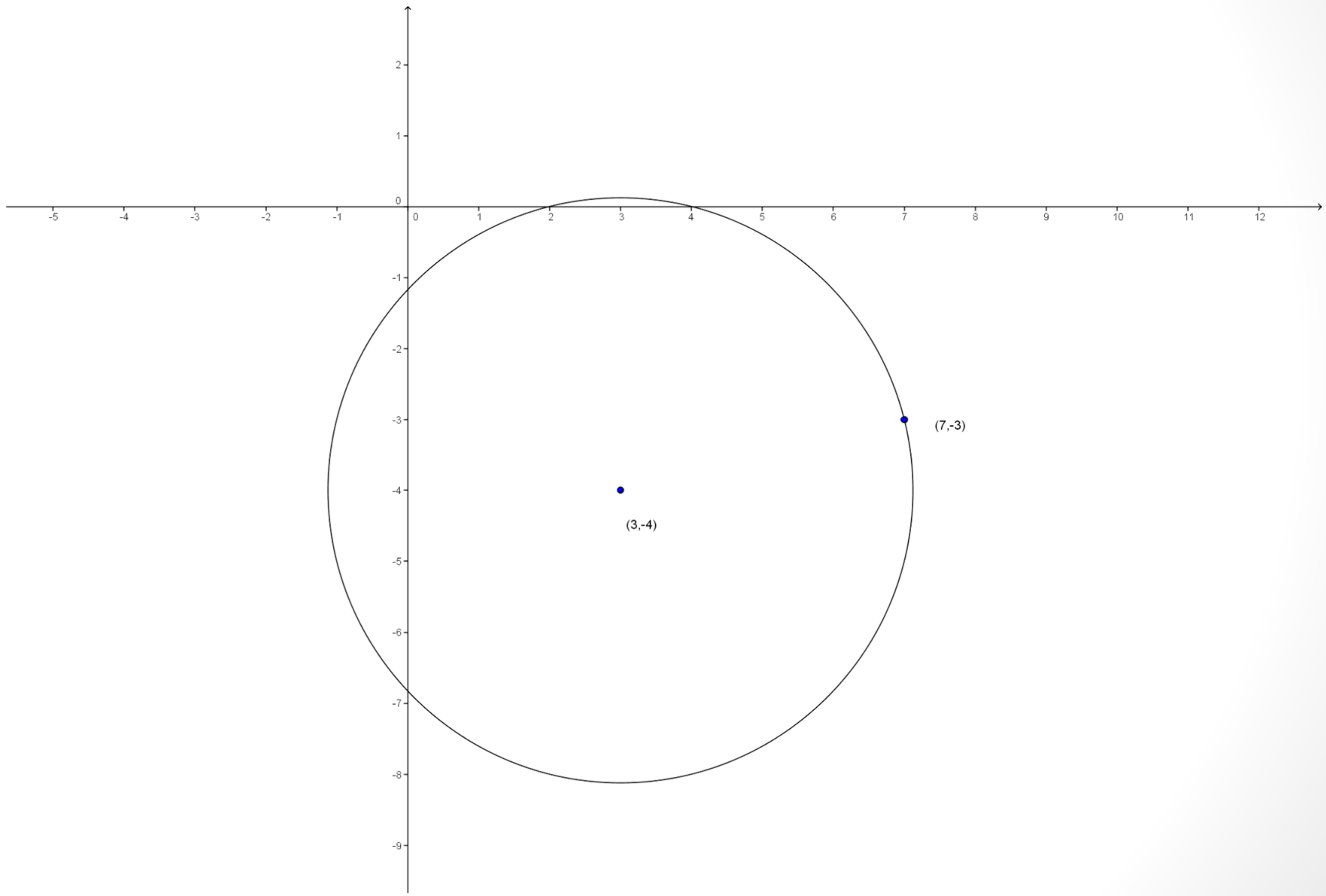
The Circle 2010

- (a) A circle with centre $(3, -4)$ passes through the point $(7, -3)$. Find the equation of the circle.

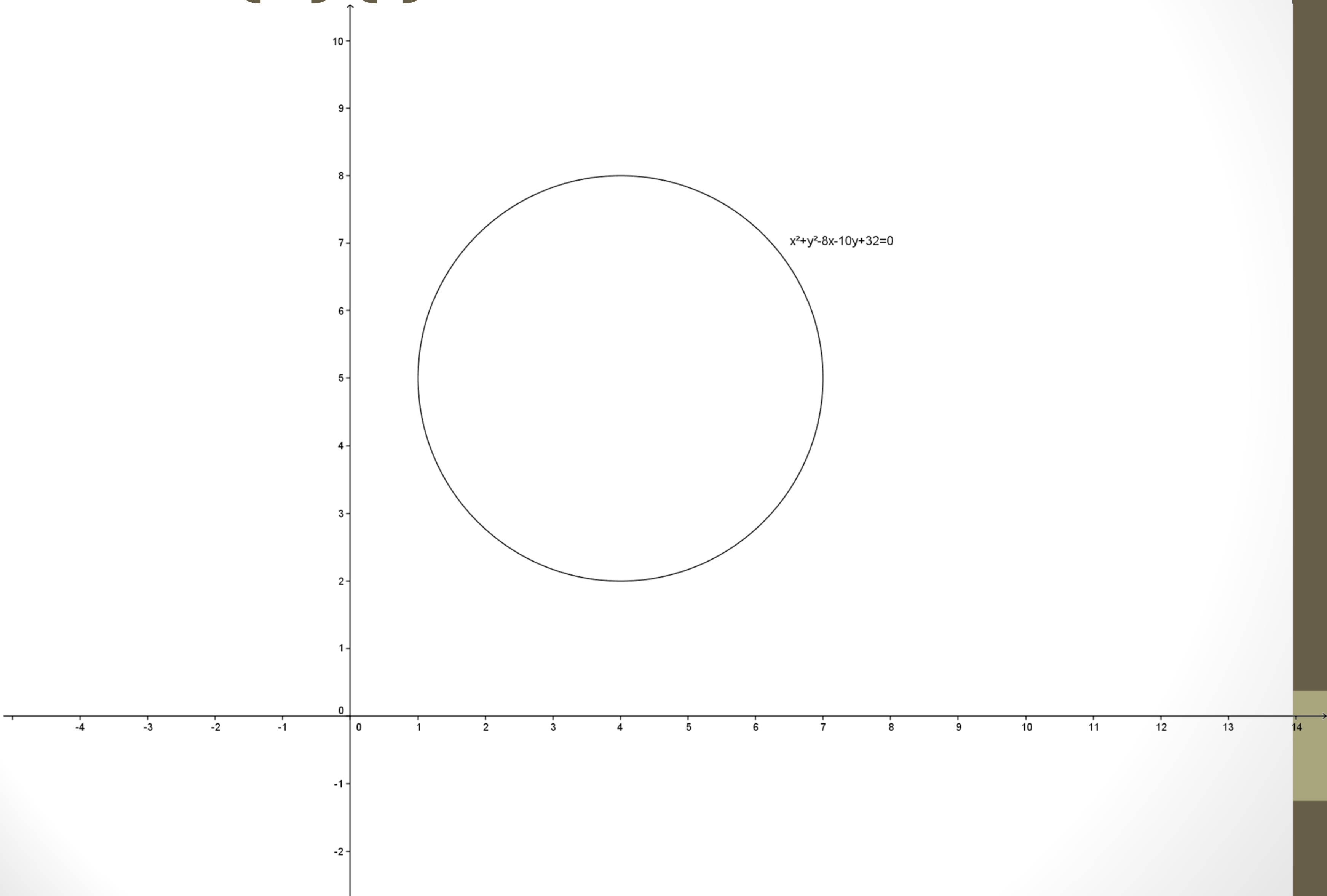
- (b)(i) Find the centre and radius of the circle $x^2 + y^2 - 8x - 10y + 32 = 0$.
- (ii) The line $3x + 4y + k = 0$ is a tangent to the circle $x^2 + y^2 - 8x - 10y + 32 = 0$. Find the two possible values of k.

- (c) A circle has the line $y = 2x$ as a tangent at the point $(2,4)$. The circle also passes through the point $(4, -2)$. Find the equation of the circle.

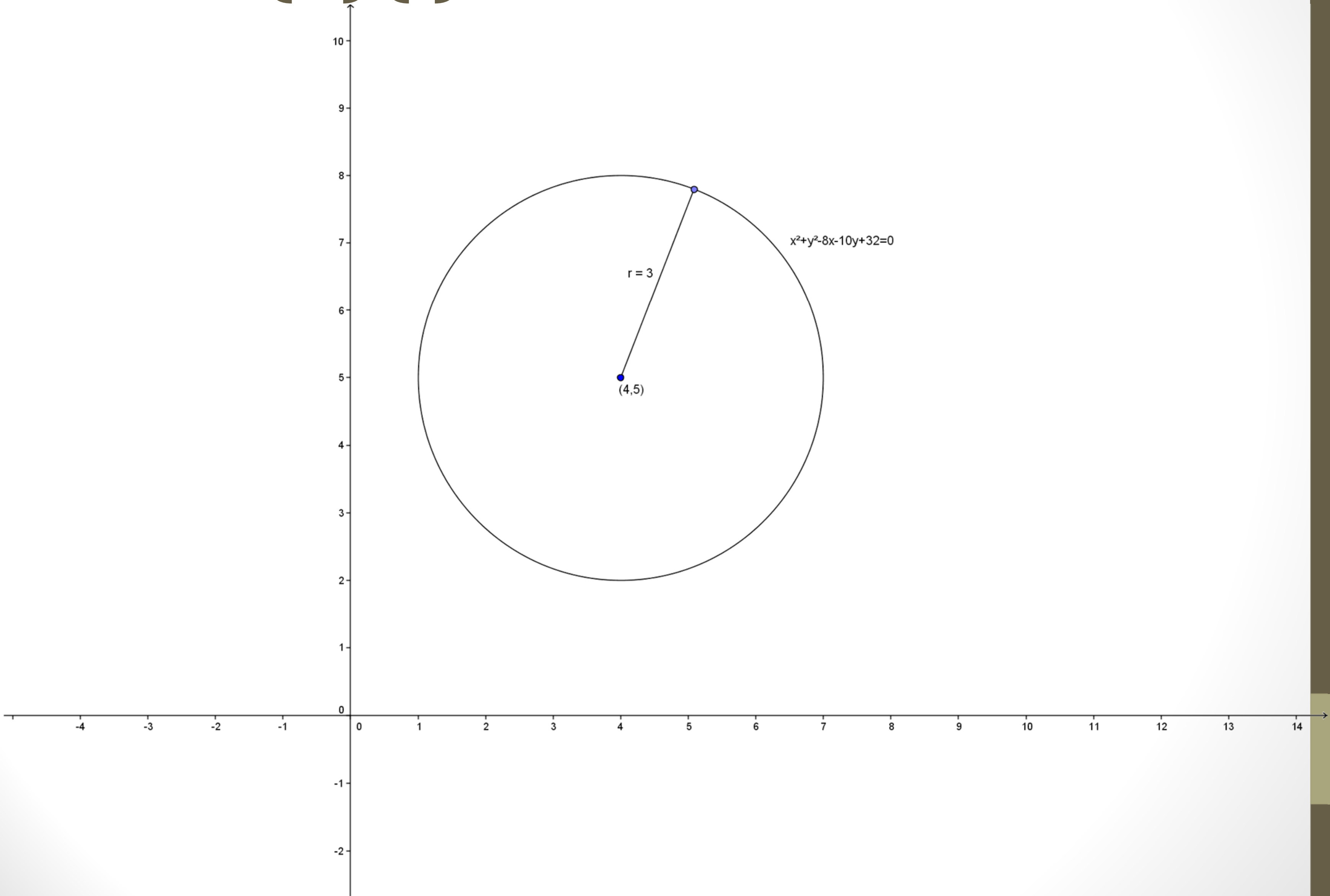
2010 (a)



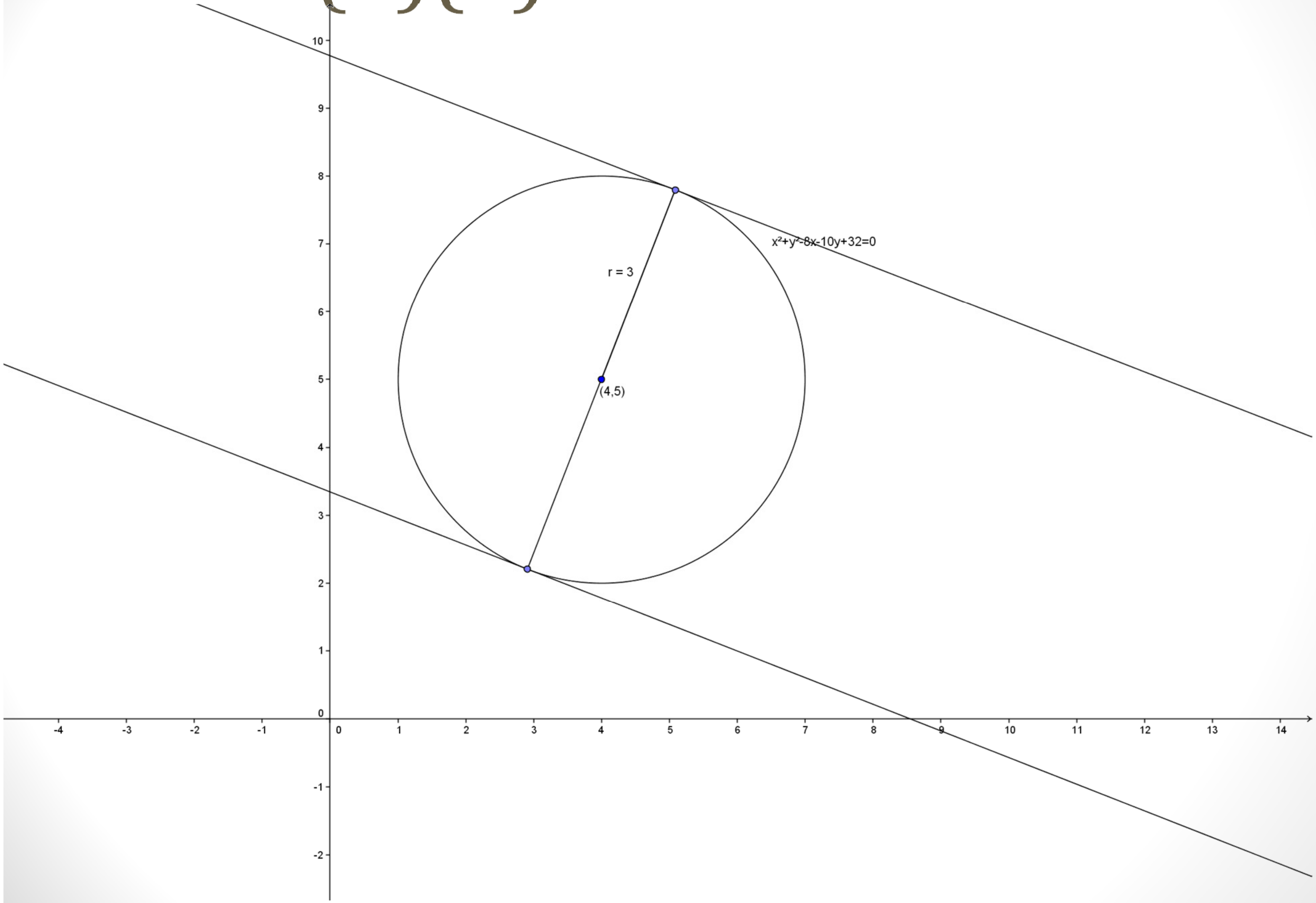
2010(b)(i)



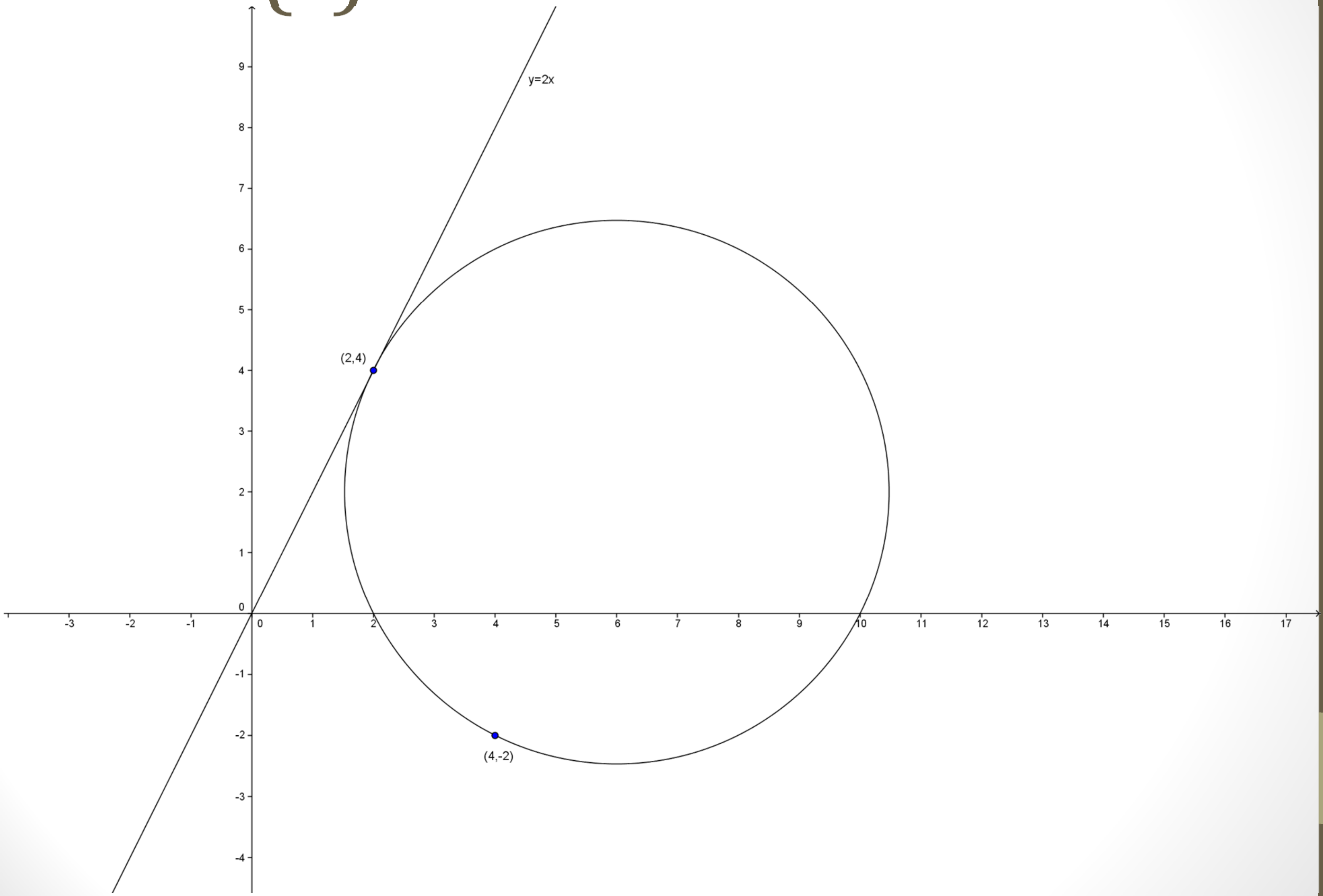
2010 (b)(i)



2010 (b)(ii)



2010 (c)



2010 (c)