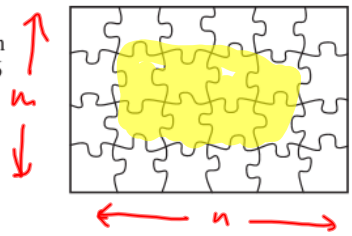
Question 6 **(50 marks)**

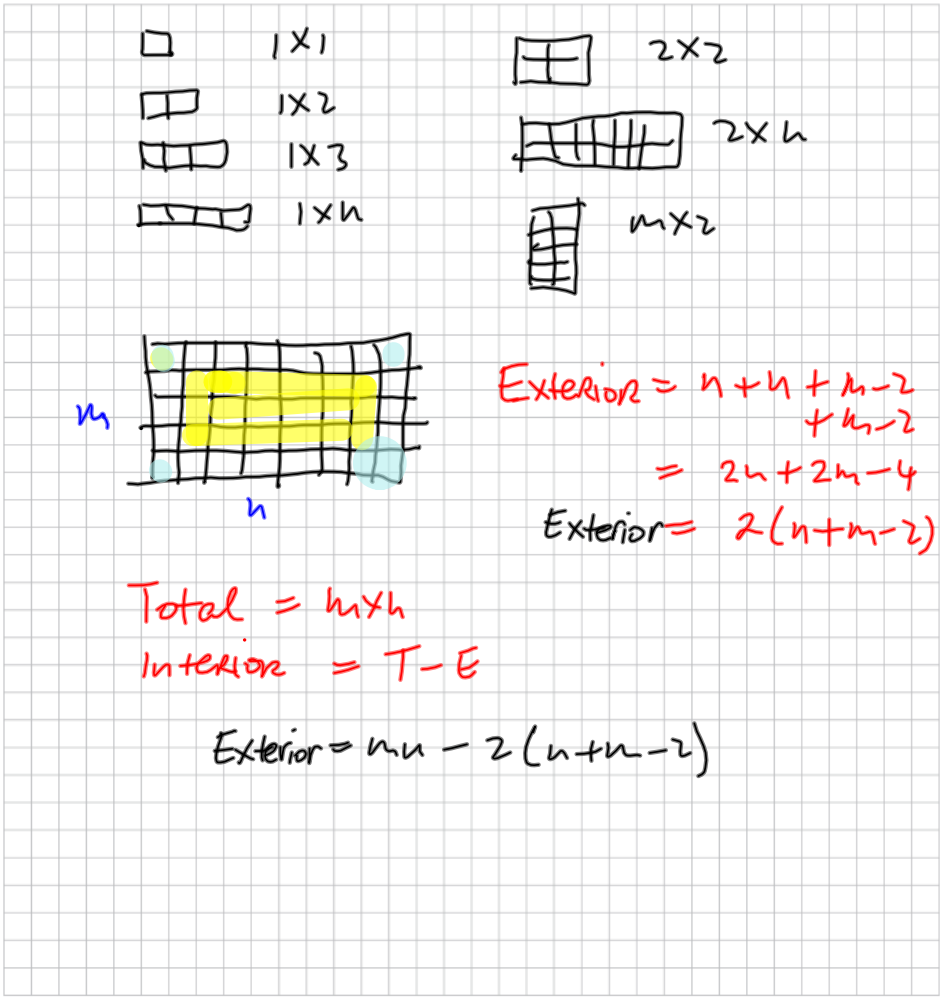
A rectangular jigsaw puzzle has pieces arranged in rows. Each row has the same number of pieces. For example, the picture on the right shows a 4×6 jigsaw puzzle – there are four rows with 6 pieces in each row.



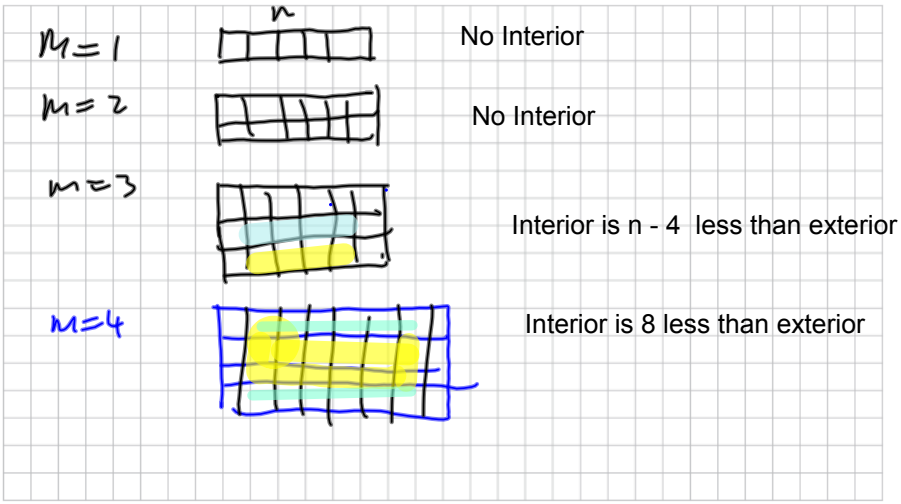
Every piece of the puzzle is either an *edge piece* or an *interior piece*. The puzzle shown has 16 edge pieces and 8 interior pieces.

Investigate the number of edge pieces and the number of interior pieces in an $m \times n$ jigsaw puzzle, for different values of m and n. Start by exploring some particular cases, and then attempt to answer the questions that follow, with justification.

Initial exploration:



(a) How do the number of edge pieces and the number of interior pieces compare in cases where either $m \leq 4$ or $n \leq 4$?



(b) Show that if the number of edge pieces is equal to the number of interior pieces, then

$$m = 4 + \frac{8}{n-4}.$$

exterior $= 2(m+n-2)$

interior $= mn - 2(m+n-2)$

$\Rightarrow \quad 2(m+n-2) = mn - 2(m+n-2)$

$4(m+n-2) = mn$

$4m + 4n - 8 = mn$

$4n - 8 = mn - 4m$

$4n - 8 = m(n-4)$

$m = \frac{4n-8}{n-4}$

$= \frac{4n-16+8}{n-4} = \frac{4(n-4)}{n-4} + \frac{8}{n-4}$

$= 4 + \frac{8}{n-4}$

(c) Find all cases in which number of edge pieces is equal to the number of interior pieces.

We know $m < 4$, $n < 4$

$$m = 4 + \frac{8}{n-4}$$

$m \times n$

let $n = 5 \Rightarrow m = 4 + \frac{8}{5-4} = 4 + 8 = 12 \Rightarrow 12 \times 5$

$n = 6 \Rightarrow m = 4 + \frac{8}{6-4} = 4 + 4 = 8 \Rightarrow 8 \times 6$

$n = 7 \Rightarrow m = 4 + \frac{8}{7-4} = 4 + \frac{8}{3}$ not possible $m \in \mathbb{N}$

$n = 8 \Rightarrow m = 4 + \frac{8}{8-4} = 4 + 2 = 6 \Rightarrow 6 \times 8$

skip $n = 9, 10, 11$ as $m \in \mathbb{N}$

$n = 12 \Rightarrow m = 4 + \frac{8}{12-4} = 4 + 1 = 5 \Rightarrow 5 \times 12$

(d) Determine the circumstances in which there are *fewer* interior pieces than edge pieces. Describe fully all such cases.

As already discovered if $m \leq 4$ or $n \leq 4$ there are fewer interior.

If exterior greater $\Rightarrow 2(m+n-2) > mn - 2(m+n-2)$

$$\Rightarrow m > 4 + \frac{8}{n-4}$$

$$n > 4 + \frac{8}{m-4}$$

by OR symmetry