Trigonometric identities

Key Words

identity **unit circle** **sine rule** **cosine rule** **compound angle** **surd form** **double angle** **half-angle formulae** **product formulae** **$\sin^{-1} x$ (arc sine x)**

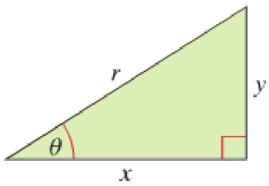
We are already familiar with the three basic trigonometric ratios, namely sine, cosine and tangent.

Three related ratios are defined as:

$$\operatorname{cosec} A = \frac{1}{\sin A} \qquad \sec A = \frac{1}{\cos A} \qquad \cot A = \frac{1}{\tan A}$$

In the given triangle

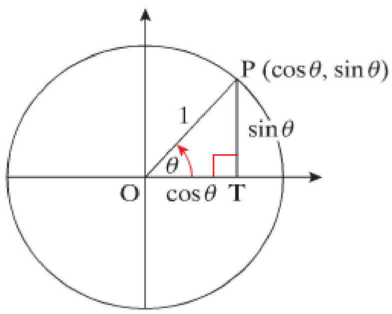
$$\frac{\sin \theta}{\cos \theta} = \frac{y}{r} \div \frac{x}{r} = \frac{y}{r} \times \frac{r}{x} = \frac{y}{x} = \tan \theta$$



$$\Rightarrow \quad \mathbf{\tan \theta = \frac{\sin \theta}{\cos \theta}}$$

This relationship between trigonometric ratios is called an **identity** because it is true for **all values** of θ.

We have already established that any point on the **unit circle** is defined by the coordinates **($\cos \theta, \sin \theta$)**.



In the given diagram $|OP| = 1$

$\Rightarrow \quad |OP|^2 = 1$

$\Rightarrow \quad \sqrt{(\cos \theta - 0)^2 + (\sin \theta - 0)^2} = 1$

$\Rightarrow \quad \sqrt{\cos^2 \theta + \sin^2 \theta} = 1$

$\Rightarrow \quad \mathbf{\cos^2 \theta + \sin^2 \theta = 1}$ ... (squaring both sides) $*$

If each term in the equation $\sin^2 \theta + \cos^2 \theta = 1$ is divided by $\cos^2 \theta$, we get,

$$\frac{\sin^2 \theta}{\cos^2 \theta} + \frac{\cos^2 \theta}{\cos^2 \theta} = \frac{1}{\cos^2 \theta}$$

$\Rightarrow \quad \tan^2 \theta + 1 = \sec^2 \theta$

$\Rightarrow \quad \mathbf{1 + \tan^2 \theta = \sec^2 \theta}$

The identities established above should be memorised as they are used very frequently to prove more complex identities. These identities are highlighted in the box below.

1. $\text{cosec } \theta = \dfrac{1}{\sin \theta}$	**2.** $\sec \theta = \dfrac{1}{\cos \theta}$	**3.** $\tan \theta = \dfrac{\sin \theta}{\cos \theta}$
4. $\cot \theta = \dfrac{\cos \theta}{\sin \theta}$	**5.** $\sin^2 \theta + \cos^2 \theta = 1$	**6.** $1 + \tan^2 \theta = \sec^2 \theta$

It follows from **5** that $\sin^2 \theta = 1 - \cos^2 \theta$ and $\cos^2 \theta = 1 - \sin^2 \theta$.

The general method of proving an identity is to choose the left-hand side and show, by using known identities, that it can be simplified into the form of the right-hand side.

This is illustrated in the following examples.

Example 1

Prove these identities:

(i) $\sec A - \tan A \sin A = \cos A$ (ii) $\tan \theta \sqrt{1 - \sin^2 \theta} = \sin \theta$.

$\text{Sec} A = \dfrac{1}{\cos A}$

$\text{Tan} A = \dfrac{\sin A}{\cos A}$

$\sin^2 A + \cos^2 A = 1$

$\Rightarrow \cos^2 A = 1 - \sin^2 A$

(i)

$$\text{LHS} = \frac{1}{\cos A} - \left(\frac{\sin A}{\cos A}\right) \sin A$$

$$= \frac{1 - \sin^2 A}{\cos A} = \frac{\cos^2 A}{\cos A} = \cos A$$

(ii)

$$\text{LHS} = \left(\frac{\sin\theta}{\cos\theta}\right)\left(\sqrt{\cos^2\theta}\right)$$

$$= \frac{\sin\theta}{\cancel{\cos\theta}}(\cancel{\cos\theta}) = \sin\theta$$

Example 2

Prove that $\dfrac{\tan\theta + \sin\theta}{\sec\theta + 1} = \sin\theta$.

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

$\sec\theta = \dfrac{1}{\cos\theta}$

$$LHS = \frac{\left(\frac{\sin\theta}{\cos\theta} + \sin\theta\right)}{\left(\frac{1}{\cos\theta} + 1\right)} \frac{(\cos\theta)}{(\cos\theta)}$$

$$= \frac{\sin\theta + \sin\theta\cos\theta}{1 + \cos\theta}$$

$$= \frac{\sin\theta\,\cancel{(1+\cos\theta)}}{\cancel{(1+\cos\theta)}} = \sin\theta$$

Identities involving the *Sine Rule* and *Cosine Rule*

The *Sine Rule* states that $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$

This can be also written as $\sin A = \dfrac{a\sin B}{b}$

The *Cosine Rule* states that $a^2 = b^2 + c^2 - 2bc\cos A$.

This can also be written as $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$.

Identities involving the sides a, b and c of a triangle generally require the use of the *Sine* or *Cosine Rules* to prove them.

Example 3

Prove that in any triangle, $c \cos B - b \cos C = \dfrac{c^2 - b^2}{a}$

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$\cos A = \frac{a^2 - b^2 - c^2}{-2bc}$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$LHS = \cancel{c}\left(\frac{a^2 + c^2 - b^2}{2a\cancel{c}}\right) - \cancel{b}\left(\frac{b^2 + a^2 - c^2}{2a\cancel{b}}\right)$$

$$= \frac{\cancel{a^2} + c^2 - b^2 - b^2 - \cancel{a^2} + c^2}{2a}$$

$$= \frac{2c^2 - 2b^2}{2a} = \frac{c^2 - b^2}{a}$$

* HW - p.404 Q1-6